Abstract Interpretation

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Abstract interpretation

- Theoretical framework unifying different program analyses
- Formal underpinning of sound and correct static analyses

- Practice: Code Contracts, Astree, Polyspace
Reusing concepts from static analysis

- Control-flow graphs
- Finite lattices
- Transfer functions
- Fixed points
  - Iterative computation
  - Work list algorithm
New concepts

- Explicit abstraction
- Concrete domain
- Abstract domain
- Galois connections

**Purpose: constructing sound abstractions**
Example program

```c
int compute(int x, int y) {
    Requires (x >= 0 && y >= 0);
    y = y + 1;
    z = x + y;
    Assert (z >= 1);
    return z;
}
```
Concrete domain

- Finite lattice $C = (E_C, \sqsubseteq^C)$
  - Set of concrete elements $E_C$
  - Partial order $\sqsubseteq^C$ on $E_C$

- Notation abuse
  - Symbol $C$ means both the finite lattice and the set of concrete elements

- Example
  - Possible values of an integer variable
  - $E_C = 2^\mathbb{N}$ (all possible subsets)
  - $\sqsubseteq^C = \subseteq$ (plain subset ordering)
Abstract domain

- Finite lattice $A = (E_A, \sqsubseteq^A, \bot, \top, \sqcup, \sqcap)$
  - Set of abstract elements $E_A$
  - Partial order $\sqsubseteq^A$ on $E_A$
  - Least abstract element $\bot$
  - Greatest abstract element $\top$
  - Join operator $\sqcup$
  - Meet operator $\sqcap$
Abstract domain: Intervals

- Definition
  - $E_A = \{ [x, y] \mid x, y \in \mathbb{Z} \cup \{-\infty, +\infty\} \}$
  - Partial order $\sqsubseteq^A$: interval inclusion
  - $\bot$ is empty interval
  - $\top = \{-\infty, +\infty\}$

- Examples
  - $[0, 2] \sqsubseteq^A [0, 4]$
  - $[0, 2] \not\sqsubseteq^A [1, 3]$
Relation between domains

- Abstraction function $\alpha : C \mapsto A$
  - Computes the most precise abstract representation

- Concretization function $\gamma : A \mapsto C$

- Example: interval domain
  - $\alpha(S) = [\min(S), \max(S)]$, $S = \{s_1, ..., s_N\}$
  - $\gamma([u, v]) = \{ x \in \mathbb{Z} \mid u \leq x \leq v \}$
Galois connection

- **Necessary conditions**
  - Both functions $\alpha$ and $\gamma$ are monotone
  - $\forall a \in A, c \in C : \alpha(c) \sqsubseteq^A a \iff c \sqsubseteq^C \gamma(a)$

- **Relation between partially ordered sets $A$ and $C$**

- **Characterizes sound abstraction**
  - We can lose precision (over-approximating)
    $$c \sqsubseteq^C \gamma \circ \alpha(c) \quad \alpha \circ \gamma(a) \sqsubseteq^A a$$
Transfer functions

- Goal: represent effects of program statements

- Concrete transfer function $\tau_C : C \mapsto C$
  - Expresses concrete semantics of program statements

- Abstract transfer function $\tau_A : A \mapsto A$
  - Expresses abstract semantics of program statements

- Relation: $\forall a \in A : \tau_C \circ \gamma(a) \subseteq \gamma \circ \tau_A(a)$

- Concrete program $P_C$
- Abstract program $P_A$
How to compute solution

- Input problem
  - Concrete program $P_C$
  - Abstract domain $A$
  - Functions $\alpha$ and $\gamma$
  - Transfer function $\tau_A$

- Representing information
  - Separate analysis value for each program variable
  - One large set with values for all program variables
How to compute solution

- Input analysis problem
- Representing information

- Approach: find \( lfp(P_A) \)
  - Symbol \( lfp \sim \) least fixed point
  - Using the work-list algorithm

- Result: \( lfp(P_C) \preceq^C \gamma(lfp(P_A)) \)
Problem: **fixpoint computation may diverge**
- Why: infinite increasing chains (sequences)

**Ascending Chain Condition (ACC)**
- Strictly ascending sequence of elements reaches some fixed point (terminates)
- Example: \( a_1 \subseteq a_2 \subseteq a_3 \subseteq \ldots \subseteq a_n = a_{n+1} = a_{n+2} \)

**Solution:** **Widening operator**
Widening

- Widening operator $\nabla : A \times A \mapsto A$
  - $\forall a_1, a_2 \in A : (a_1 \sqsubseteq a_1 \nabla a_2) \land (a_2 \sqsubseteq a_1 \nabla a_2)$

- Sequence $w_0 = a_0$, ..., $w_{i+1} = w_i \nabla a_{i+1}$ not strictly increasing
  - fixed point computation will terminate

- Other benefit: faster convergence
“Intervals” does not satisfy ACC

Option 1
- Keep stable bounds (preserve the value)
- Extrapolate unstable bounds (${-\infty, +\infty}$)

Option 2
- Keep valid bounds from the first operand
- Extrapolate bounds otherwise (${-\infty, +\infty}$)
• Consequence: losing precision

• Practice
  ▪ Use widening operator
    ▪ on backward edges in CFG
    ▪ for really too big intervals

• Remedy: Narrowing
  ▪ Complementary operator
  ▪ Goal: improving precision
Numerical abstract domains

- Non-relational domains
  - Program variables treated separately
  - Examples: signs, intervals

- Relational domains
  - Consider relations between variables
  - Example: predicate abstraction
Cartesian abstraction

- Very important special case

- Key idea
  - $\alpha$: flattening the analysis information
  - $\gamma$: restores all possible combinations

- Pros: better scalability
- Cons: loses precision

- Example: predicate abstraction
Other abstract domains

- **Octagon**
  - Values represented as constraints $\pm x \pm y \leq c$

- **Polyhedral**
  - Values represented as constraints $\sum_i a_{ij} * x_i \leq c_j$

- **Linear equations**
Using abstract interpretation

1) Design the abstract domains

2) Define abstraction functions

3) Design widening operators

4) Define all transfer functions
Multiple abstract domains

- **Combination**
  - Two abstract domains $A_1, A_2$
  - Cartesian product: $A_1 \times A_2$
  - $\forall <a_1,a_2> \in A_1 \times A_2 : \gamma_{A_1 \times A_2}(<a_1,a_2>) \subseteq (\gamma_{A_1}(a_1) \cap \gamma_{A_2}(a_2))$

- **Composition**
  - One concrete domain $C$
  - Abstract domains $A_1, A_2$
  - Galois connection
    - $\alpha_1, \gamma_1$ between $C$ and $A_1$
    - $\alpha_2, \gamma_2$ between $A_1$ and $A_2$
  - We get the connection between $C$ and $A_2$
    - Functions: $\alpha_2 \circ \alpha_1, \gamma_1 \circ \gamma_2$
Tools

- Clousot
  - Program analyzer for Code Contracts (C#/.NET)
  - Verifies method contracts and low-level errors

- Astrée
  - Static analyzer for programs written in C
    - Programs without dynamic memory allocation and recursion
  - Industrial applications (Airbus A340 SW)

- Polyspace
  - Static analysis toolset for programs in C/C++/Ada
Further reading

