7. Timed Automata
Timed Automata
**System model**

AG (start $\rightarrow$ AF heat)

Property specification

**Model Checker**

Property satisfied

Property violated
TIMED_AUTOMATA

System model

Timed CTL

AG (start $\rightarrow$ AF heat)

Property specification

Model Checking

Property satisfied

Property violated
Büchi automaton is finite automaton accepting infinite words
Word is accepted if:
- An accepting state is visited infinitely many times (standard case)
- A state from each accepting set is visited infinitely many times (generalized case)

$L_{BA} = (a + b)^* a^\omega$
Timed Languages

Timed sequence $t = t_1 t_2 t_3 ...$ is infinite sequence of values $t_i \in \mathbb{R}^+$ satisfying:

- monotonicity: $\forall i \geq 1 : t_i < t_{i+1}$
- progress: $\forall x \in \mathbb{R} : \exists i \geq 1 : t_i > x$

Timed word is a tuple $(s, t)$ where $s$ is infinite sequence of alphabet symbols and $t$ is timed sequence.
In addition to Büchi automaton, Timed automaton contains set of real variables representing clocks.

Each clock variable:
- is initially set to 0
- increments at the same speed as any other clock variable
- can be reset to 0 at any transition
- (co-)defines guard upon transitions

Timed automaton accepts timed language, i.e., (finite or infinite) set of timed words.
**CLOCK CONSTRAINTS**

Given set of clock \( X \), set \( \Phi(X) \) of clock constraints \( \delta \) is defined:

\[
\delta := x \leq c \mid c \leq x \mid \neg \delta \mid \delta_1 \land \delta_2
\]

where \( x \in X, \ c \in \mathbb{Q} \).
Nondeterministic timed automaton $A$ is 6-tuple $(\Sigma, S, S_0, C, E, F)$:

- $\Sigma$ is finite alphabet
- $S$ is finite set of states
- $S_0 \subseteq S$ is set of initial states
- $C$ is finite set of clocks
- $E \subseteq S \times S \times \Sigma \times 2^C \times \Phi(C)$ is transition relation
  - $2^C$ denotes the set of clocks to be reset at transition
  - $\Phi(C)$ is clock constraint over $C$
- $F \subseteq S$ is set of accepting states
Automaton accepting language:

\[\{( (abcd)^{\omega}, t) \mid \forall j : ((t_{4j+3} < t_{4j+1} + 1) \land (t_{4j+4} > t_{4j+2} + 2)) \}\]
**Question:** Is class of timed regular languages closed under **finite union**?

- Yes

**Proof:** Since the TA are nondeterministic, union is represented by disjoint union of particular automata (similar to Büchi automata).
**Properties of Timed Automata**

**Question:** Is class of timed regular languages closed under *intersection*?

- Yes

**Proof:** Simple modification of intersection of Büchi automata: Let $A$ be automaton accepting the intersection of languages of $A_1$ and $A_2$ and $C_i$ be set of clocks. Transitions of $A$ are $((s_1, s_2, i), (s'_1, s'_2, j), a, \lambda, \varphi)$:
  - $(s_1, s_2, i), (s'_1, s'_2, j), a$ as in case of intersection of two Büchi automata
  - $\lambda = \lambda_1 \cup \lambda_2$ is set of clocks to be reset
  - $\varphi = \varphi_1 \land \varphi_2$ is transition constraint

Assuming disjunct sets of clocks, states, and transitions, alphabet is shared, though.
**Question:** Is class of timed regular languages closed under **complement**?

- **No**

Even worse—inclusion of timed languages $L(A) \subseteq L(B)$ is **undecidable** problem.
Verification of properties realized by checking of language emptiness, similarly to LTL model checking.

Systematic exploration of timed automata not feasible due to infinite (usually even uncountable) number of possible clock valuations.

Idea: Constructing “equivalent” Büchi automaton accepting the same language up to timing as original timed automaton.
- corresponding Büchi automaton is called region automaton.
States of region automaton are regions.

Each region corresponds to state and set of equivalent clock valuations of original timed automaton.

Transformation to region automaton solves the problem of uncountable many clock valuations disallowing systematic exploration of state space.
Given timed automaton $A$, $(s, n)$ denotes *extended state*

- $s$ is state of $A$
- $n$ is clock interpretation (i.e., valuation of clock variables)

For $t \in \mathbb{R}$: $t = \lfloor t \rfloor + \text{fract}(t)$. 
Let $A = (\Sigma, S, S_0, C, E, F)$ be timed automaton.

For $x \in C$, by $c_x$ denote largest $c$ such that $x \leq c$ or $c \leq x$ is sub-formula of some clock constraints in $F$

Clock valuations $n, n'$ are equivalent ($n \sim n'$) iff:

1. $\forall x \in C : \lfloor n(x) \rfloor = \lfloor n'(x) \rfloor \lor (\lfloor n(x) \rfloor > c_x \land \lfloor n'(x) \rfloor > c_x)$ and
2. $\forall x, y \in C : n(x) \leq c_x \land n(y) \leq c_y \implies fract(n(x)) \leq fract(n(y)) \iff fract(n'(x)) \leq fract(n'(y))$ and
3. $\forall x \in C : \lfloor n(x) \rfloor \leq c_x \implies fract(n(x)) = 0 \iff fract(n'(x)) = 0$

Clock region for $A$ is equivalence class induced by $\sim$. 
EXAMPLE OF CLOCK REGIONS
Example of Clock Regions

6 corner regions
EXAMPLE OF CLOCK REGIONS

6 corner regions

14 open line segments
EXAMPLE OF CLOCK REGIONS

6 corner regions
14 open line segments
8 open regions
Clock region $b$ is successor of clock region $a$ ($b \in \text{succ}(a)$) iff
\[ \forall n \in a : \exists t \in \mathbb{R}^+ : n + t \in b. \]

Successor regions are those that can be reached by incrementing time, including resetting clocks.
REGION SUCCESSORS

1. \( \forall x \in C : x > x_c \implies \text{succ}(a) = \{a\} \)

2. Let \( C_0 \) be set of clocks such that \( (x = c) \in a \). Values of \( x \in C \) for \( b = \text{succ}(a) \) satisfy:
   - if \( x = c \), then \( b \) satisfies \( x > x_c \), otherwise \( c < x < c + 1 \).
   - if \( x \notin C_0 \), constraint for \( x \) in \( a \) equals to the one in \( b \).

3. Let \( C_1 \) be set of clocks such that region \( a \) does not satisfy \( x > c \) and \( \forall y \in C_1 : \text{fract}(y) \leq \text{fract}(x) \). Define clock region \( b \) as:
   - \( (\forall x \in C_1 : (c - 1 < x < c) \in a \implies (x = c) \in b) \wedge \)
     \( (\forall x \notin C_1 : \text{constraint for } x \text{ in } a \text{ equals to the one in } b) \).
   - \( \forall x, y \notin C_1 : \text{fract}(x) \leq \text{fract}(y) \text{ in } a \iff \text{fract}(x) \leq \text{fract}(y) \text{ in } b \).

\( \text{succ}(a) = \{a, b, \text{succ}(b)\} \)
For timed automaton $A = (\Sigma, S, S_0, C, E, F)$, corresponding region automaton $R(A)$ is defined as follows:

- States of $R(A)$ are $(s, a)$ where $s \in S$ and $a$ is clock region.
- Initial states are $(s_0, [n_0])$ where $s_0 \in S_0$ and $n_0(x) = 0$ for $x \in C$.
- $((s, a), (s', a'), m)$ is transition of $R(A)$ iff $\exists (s, s', m, \lambda, \varphi) \in E$ and there exists region $a''$ such that:
  - $a''$ is successor of $a$
  - $a''$ satisfies $\varphi$
  - $a' = [\lambda \rightarrow 0]a''$
Lemma: If $r$ is progressive run of $R(A)$ over $s$, then there exists time sequence $t$ and run $r'$ of $A$ over $(s, t)$ such that $r = [r']$.

- progressive—no bound for any clock
- w.l.o.g. we can assume just progressive runs
- $[r]$ means “untiming” $r$

Theorem: For timed automaton $A$, there exists Büchi automaton $R(A)$ that accepts $\text{Untime}(L(A))$.

Idea: Construct region automaton $R(A)$ with accepting states $\{(s, a) | s \in F\}$. $R(A)$ is special case of Büchi automaton.
NETWORK OF TIMED AUTOMATA

- For modelling communicating parts of system in independent way
- Each part represented by a single TA, which Communicates with other parts through input/output actions
- Composition realized by parallel synchronous product

UPPAAL

- Tool for verification of TA models
- Academic, but quite well established and used in industry nowadays
- Supports modelling, verification, simulation
- Successfully applied on communication protocols, multimedia applications, ...
- Available at http://www.uppaal.org/ and http://www.uppaal.com