NSWI101: System Behaviour Models And Verification
10. Stochastic Model Checking

Jan Kofroň

FACULTY
OF MATHEMATICS
AND PHYSICS
Charles University

Department of
Distributed and
Dependable
Systems

MOTIVATION

In some cases **absolute** absence of errors is infeasible

- failures of particular parts of system
- non-deterministic behaviour of users
- …

It might be useful to determine level of reliability in terms of probability

- frequency of errors
- time to recovery
- throughput
- mean waiting time
- …

TODAY

Stochastic Model Checking

Lecture based on M. Kwiatkowska et al.: Stochastic Model Checking

http://www.prismmodelchecker.org/papers/sfm07.pdf

Stochastic model checking

- Not only validity of certain properties
  - but also probability of reaching states/paths
- → Need for special language
  - PCTL = Probabilistic Computational Tree Logic
  - CSL = Continuous Stochastic Logic
- **Discrete-time Markov Chains (DTMC)** are used as models for discrete time analysis
- **Continuous-time Markov Chains (CTMC)** are used for continuous time analysis

Discrete-time Markov chains

Discrete-time Markov chains

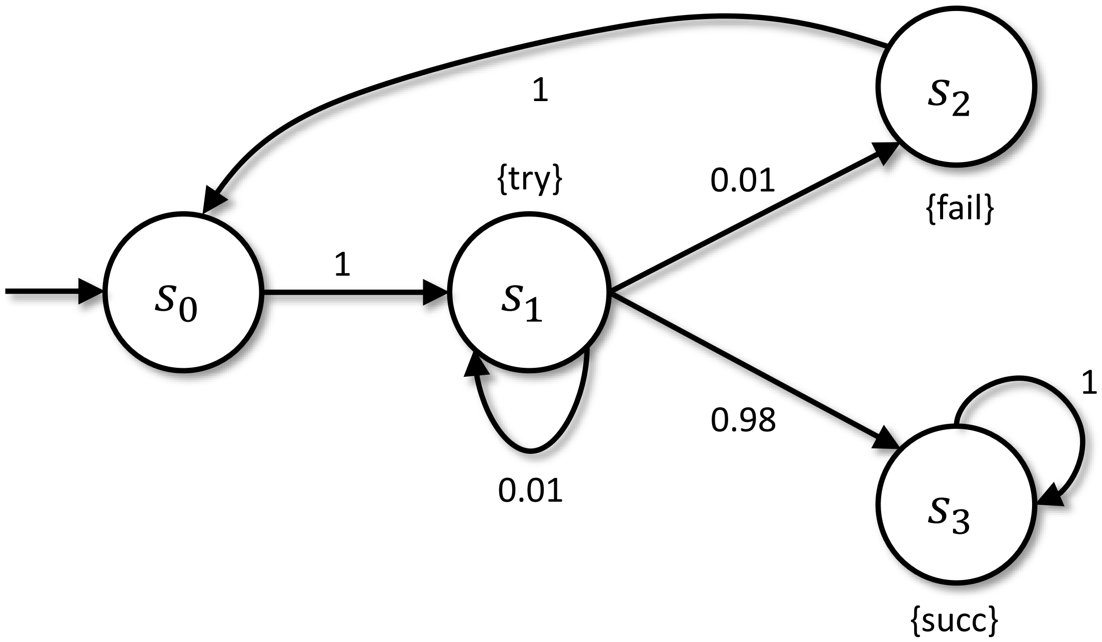
Definition: A labelled DTMC D is a tuple $(S, \bar{s}, \mathbf{P}, L)$ where:

- S is finite set of states
- $\bar{s} \in S$ is initial state
- $\mathbf{P}: S \times S \rightarrow [0,1]$ is **transition probability matrix** where $\sum_{s' \in S} \mathbf{P}(s, s') = 1$ for all $s \in S$
- $L: S \rightarrow 2^{AP}$ is labelling function assigning to each state set $L(s)$ of atomic propositions

Discrete-time Markov chains

- Sum of probabilities of transitions originating in each state must be 1!
- Terminating states can be modelled by self-loop with probability 1

EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

NOTIONS

- **Path** is non-empty sequence $s_0 s_1 s_2 \dots$ where $s_i \in S$ and $\forall i \geq 0: P(s_i, s_{i+1}) > 0$
- Path can be finite or infinite
- $Path^D(s)$ – set of **infinite** paths in D starting at s
  - this is default meaning of paths
- $Path^D_{fin}(s)$ – set of **finite** paths in D starting at s

PATH PROBABILITY

Probability for finite path $\omega_{fin} \in P^{D}_{fin}(s)$:

$$P_s(\omega_{fin}) = \begin{cases} 1 & if\ n = 0 \\ \prod_{i=0}^{n-1} P\big(\omega(i), \omega(i+1)\big) & otherwise \end{cases}$$

where n is length of ω_{fin}

Cylinder set $C(\omega_{fin}) \subseteq Path^{D}(s)$:

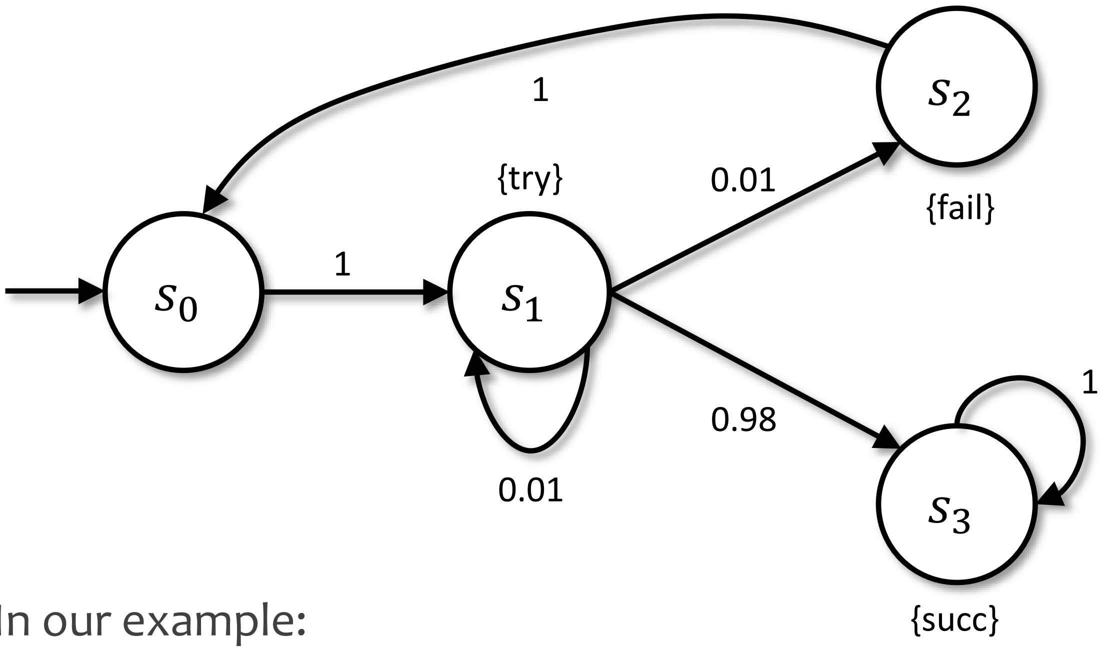
$C(\omega_{fin}) \stackrel{\text{def}}{=} \{\omega \in Path^{D}(s) |\ \omega_{fin}$ is a prefix of $\omega\}$

PROBABILITY MEASURE

Probability measure Pr_s is function defined as:

$$Pr_s\left(C(\omega_{fin})\right) = P_s(\omega_{fin}) \text{ for all } \omega_{fin} \in Path_{fin}^{D}(s)$$

PROBABILITY MEASURE – EXAMPLE



In our example:

$$Pr_{s_0}\big(C(s_0 s_1 s_1 s_1)\big) = 1.00 \cdot 0.01 \cdot 0.01 = 0.0001$$

$$Pr_{s_0}\big(C(s_0 s_1 s_1 s_2)\big) = 1.00 \cdot 0.01 \cdot 0.01 = 0.0001$$

$$Pr_{s_0}\big(C(s_0 s_1 s_1 s_3)\big) = 1.00 \cdot 0.01 \cdot 0.98 = 0.0098$$

$$Pr_{s_0}\big(C(s_0 s_1 s_2 s_0)\big) = 1.00 \cdot 0.01 \cdot 1.00 = 0.01$$

$$Pr_{s_0}\big(C(s_0 s_1 s_3 s_3)\big) = 1.00 \cdot 0.98 \cdot 1.00 = 0.98$$

Probabilistic Computational Tree Logic (PCTL)

Probabilistic Computational Tree Logic (PCTL)

- Extension of CTL
- Syntax:

  $\Phi ::= true \mid a \mid \neg\Phi \mid \Phi \wedge \Phi \mid P_{\sim p}[\phi]$ (*state formula*)

  $\phi ::= X\Phi \mid \Phi\ U^{\leq k}\ \Phi$, where (*path formula*)

  a is atomic proposition
  $\sim\ \in \{<, \leq, \geq, >\}$
  $p \in [0,1]$
  $k \in \mathbb{N} \cup \infty$

- … plus common (derived) facts:
  - $false \equiv \neg true$
  - $\Phi \vee \Psi \equiv \neg(\neg\Phi \wedge \neg\Psi)$

SEMANTICS OF PCTL

$$
\begin{aligned}
s &\vDash true && \text{for all } s \in S \\
s &\vDash a && \Leftrightarrow a \in L(s) \\
s &\vDash \neg\Phi && \Leftrightarrow s \nvDash \Phi \\
s &\vDash \Phi \wedge \Psi && \Leftrightarrow s \vDash \Phi \wedge s \vDash \Psi \\
s &\vDash P_{\sim p}[\phi] && \Leftrightarrow Prob^D(s,\phi) \sim p \\
\omega &\vDash X\Phi && \Leftrightarrow \omega(1) \vDash \Phi \\
\omega &\vDash \phi\, U^{\leq k}\, \psi && \Leftrightarrow \exists i \in \mathbb{N}: \big(i \leq k \wedge \omega(i) \vDash \psi \wedge \forall j < i: (\omega(j) \vDash \phi)\big)
\end{aligned}
$$

where $Prob^D(s,\phi) \stackrel{\text{def}}{=} Pr_s\{\omega \in Path^D(s) \mid \omega \vDash \phi\}$

Common CTL operators

CTL F and G operators:

$$P_{\sim p}[F\ \Phi] \equiv P_{\sim p}[true\ U^{\leq\infty}\Phi]$$

$$P_{\sim p}[F^{\leq k}\ \Phi] \equiv P_{\sim p}[true\ U^{\leq k}\ \Phi]$$

$$G\ \Phi \equiv\ \neg F \neg \Phi$$

$$G^{\leq k}\ \Phi \equiv\ \neg F^{\leq k} \neg \Phi$$

NEGATION

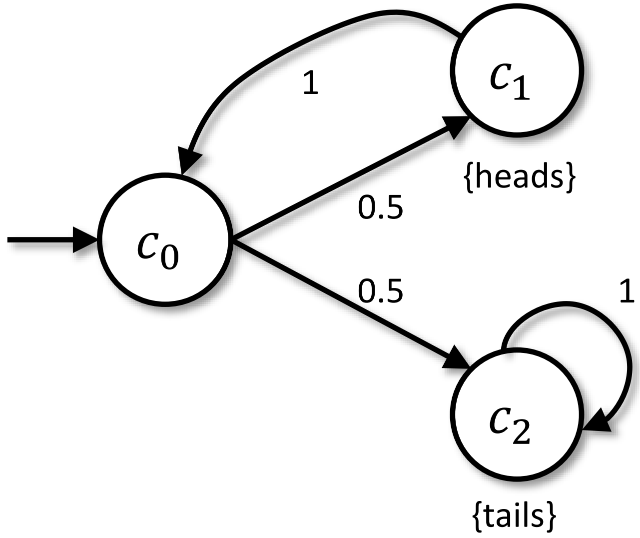
- Syntax does not allow for negation of path formulae
- However, it holds:

$$P_{\sim p}[G\ \Phi] \equiv P_{\overline{\sim}1-p}[F\ \neg\Phi]$$
$$P_{\sim p}[G^{\leq k}\ \Phi] \equiv P_{\overline{\sim}1-p}[F^{\leq k}\ \neg\Phi]$$

where $\overline{<} \equiv >, \overline{\leq} \equiv \geq, \overline{\geq} \equiv \leq, \overline{>} \equiv <$

Quantifiers

- $P_{\sim p}[\cdot]$ is probabilistic analogue to path quantifiers:
  - $EF\Phi \equiv P_{>0}[F\ \Phi]$
  - But: $AF\Phi$ is **NOT** the same as $P_{\geq 1}[F\ \Phi]$



c_0 satisfies $P_{\geq 1}[F\ tails]$
c_0 does **NOT** satisfy $AF\ tails$

Examples of PCTL properties

- $P_{\geq 0.4}[X\ delivered]$
  - probability that message gets delivered in next step is at least 0.4
- $init \rightarrow P_{\leq 0}[F\ error]$
  - error state is not reachable from any init state
- $P_{\geq 0.9}[\neg down\ U\ served]$
  - probability that server does not go down before request gets served is at least 0.9
- $P_{<0.1}[\neg done\ U^{\leq 10}\ fault]$
  - probability that error occurs before protocol is done and within 10 steps is less than 0.1

Model checking PCTL

- Based on CTL model checking algorithm
  1. decomposing formula into sub-formulae
  2. in bottom-up manner finding set of states satisfying particular sub-formulae
  3. the set of states for the input formula at root

- Special handling of the P formulae

XΦ

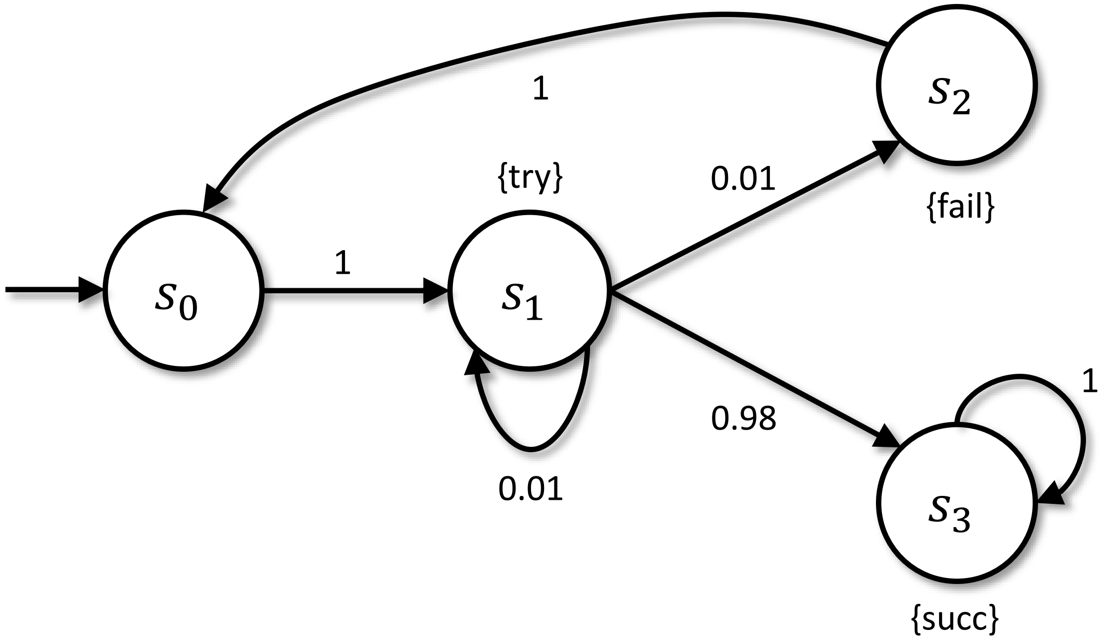
- For $P_{\sim p}[X\ \Phi]$ we need to compute $Prob^D(s, X\ \Phi)$ for each state s:

$$Prob^D(s, X\Phi) = \sum\nolimits_{s' \in Sat(\Phi)} \mathbf{P}(s, s')$$

where $Sat(\Phi)$ is set of states satisfying Φ

- Let $\underline{\Phi}(s) = \begin{cases} 1 & if\ s \in \mathrm{Sat}(\Phi) \\ 0 & otherwise \end{cases}$
- $\underline{Prob^D}(X\Phi) = \mathbf{P} \cdot \underline{\Phi}$
  - Vector with probabilities for particular states

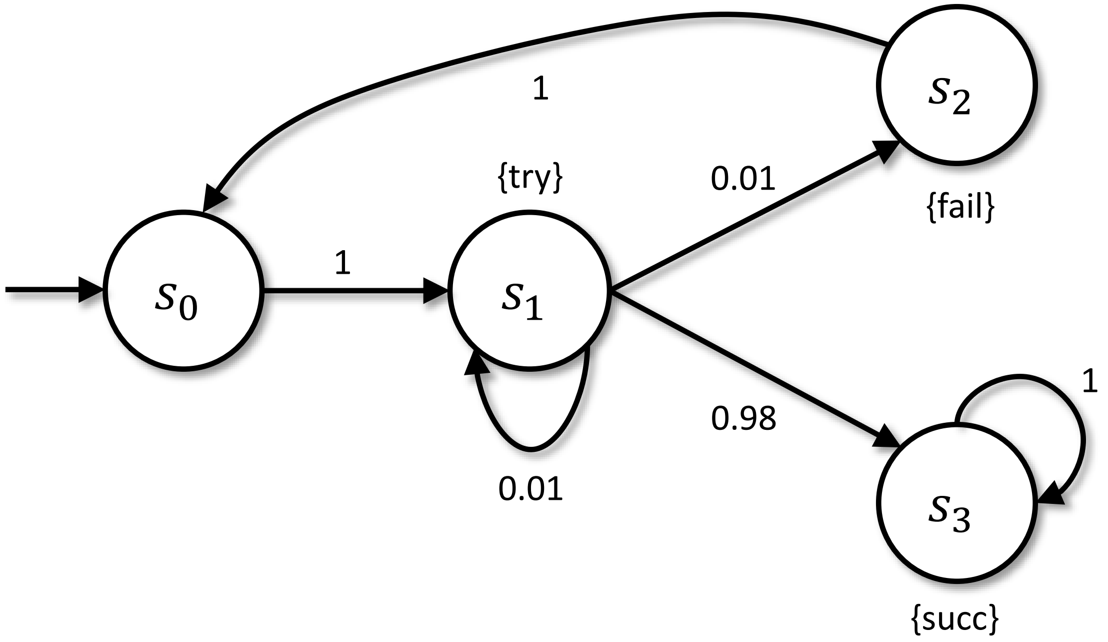
XΦ – Example



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{\geq 0.9}[X(\neg try \vee succ)]$$

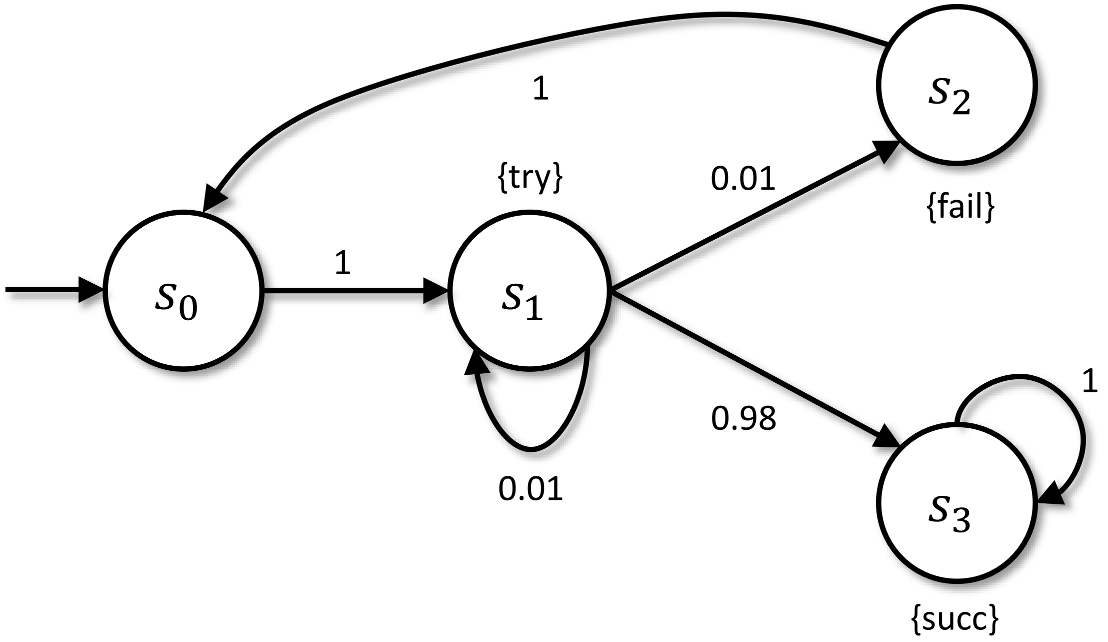
XΦ – Example



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Sat(\neg try \lor succ) = \{s_0, s_2, s_3\} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

XΦ – Example



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.99 \\ 1 \\ 1 \end{pmatrix}$$

$\mathbf{\Phi\ U^{\leq k}\ \Psi}$ – FOR $k \neq \infty$

For $P_{\sim p}[\Phi\ U^{\leq k}\ \Psi]$ we need to compute $Prob^D(s, \Phi\ U^{\leq k}\ \Psi)$ for each state s:

$$Prob^D\left(s, \Phi\ U^{\leq k}\ \Psi\right) =$$

$$= \begin{cases} 1 & if\ s \in Sat(\Psi) \\ 0 & if\ k = 0\ or\ s \in Sat(\neg\Phi \wedge \neg\Psi) \\ \sum_{s' \in S} \mathbf{P}(s, s') \cdot Prob^D\left(s', \Phi\ U^{\leq k-1}\ \Psi\right) & \text{otherwise} \end{cases}$$

where $Sat(\Phi)$ is set of states satisfying Φ

$\boldsymbol{\Phi}\ \mathbf{U}^{\leq \mathbf{k}}\ \boldsymbol{\Psi}$ – FOR $\boldsymbol{k} \neq \infty$

Definition: For any $DTMC\ D = (S, \bar{s}, \mathbf{P}, L)$ and PCTL formula Φ, let $D[\Phi] = (S, \bar{s}, \mathbf{P}[\Phi], L)$ where, if $s \not\models \Phi$, then $\mathbf{P}[\Phi](s, s') = \mathbf{P}(s, s')$ for all $s' \in S$, and if $s \models \Phi$, then $\mathbf{P}[\Phi](s, s) = 1$ and $\mathbf{P}[\Phi](s, s') = 0$ for all $s' \neq s$.

Then it holds:

$$Prob^D\left(s, \Phi\ U^{\leq k}\ \Psi\right) = \sum_{s' \models \Psi} \pi_{s,k}^{D[\neg\Phi \vee \Psi]}(s')$$

$\mathbf{\Phi\ U^{\leq k}\ \Psi}$ – FOR $\boldsymbol{k} \neq \infty$

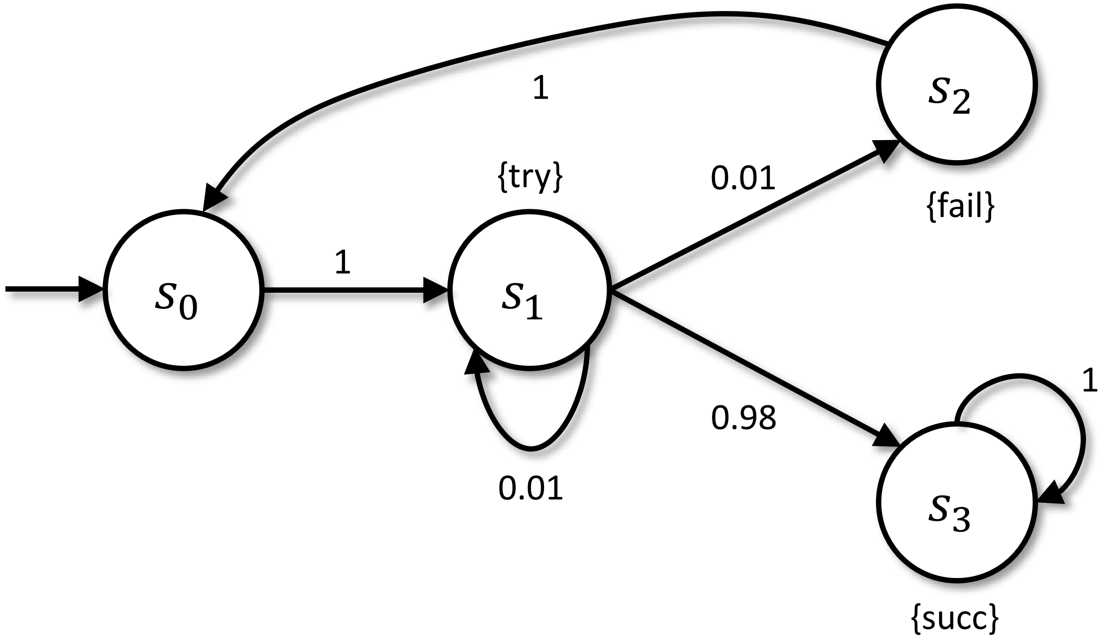
Vector of probabilities $\underline{Prob^D}(\Phi\ U^{\leq k}\ \Psi)$ can be computed as:

$$\underline{Prob^D}\left(\Phi\ U^{\leq k}\ \Psi\right) = \ (\mathbf{P}[\neg\Phi \vee \Psi])^k \cdot \underline{\Psi}$$

Usually computed in iterative way

- but can be pre-computed for particular k

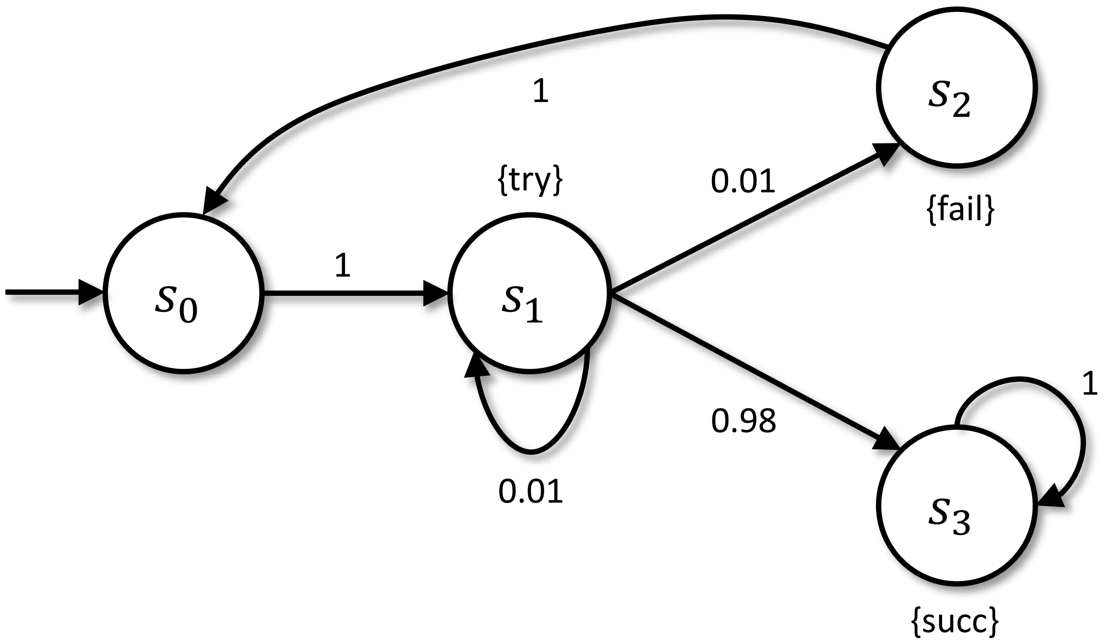
$\Phi\ U^{\leq k}\ \Psi$ – for $k \neq \infty$ – Example



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{P}_{>0.98}[F^{\leq 2} succ] = \mathbf{P}_{>0.98}[true\ U^{\leq 2} succ]$$

$\Phi\ \mathrm{U}^{\leq k}\ \Psi$ – for $k \neq \infty$ – Example

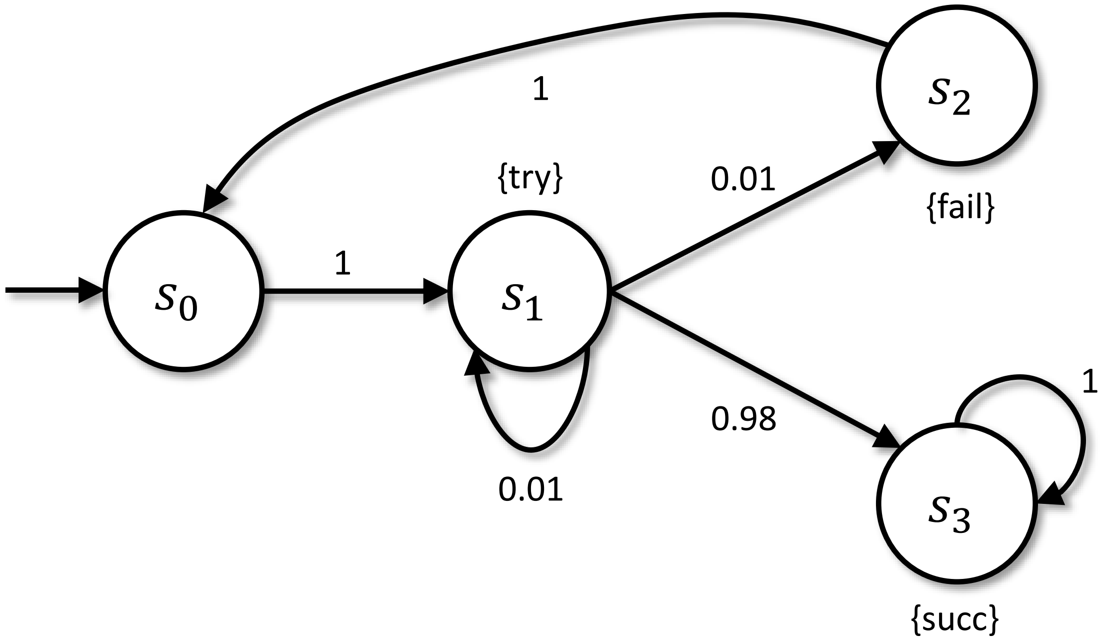


$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Sat(true) = \{s_0, s_1, s_2, s_3\}, \qquad Sat(succ) = \{s_3\}$$

$$\mathbf{P}[\neg true \lor succ] = \mathbf{P}$$

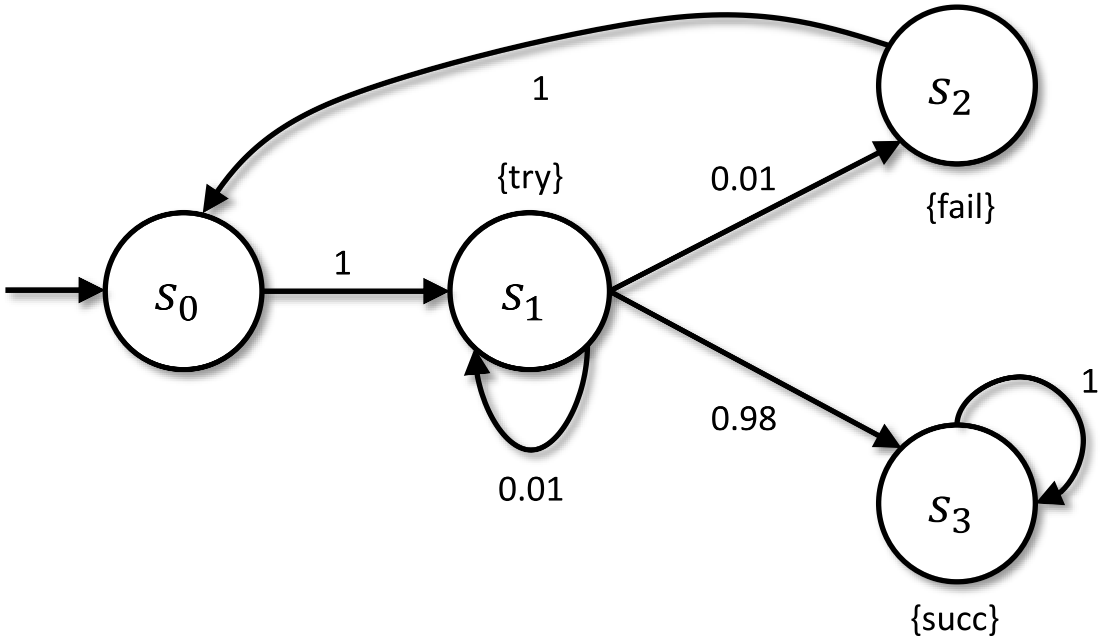
$\Phi\ \mathrm{U}^{\leq \mathrm{k}}\ \Psi$ – FOR $k \neq \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{Prob}^D(\Phi\ U^{\leq 0}\Psi) = succ = [0,0,0,1]$$
$$\underline{Prob}^D(\Phi\ U^{\leq 1}\Psi) = \mathbf{P}[\neg true \vee succ] \cdot \underline{Prob}^D(\Phi\ U^{\leq 0}\ \Psi) = [0,0.98,0,1]$$
$$\underline{Prob}^D(\Phi\ U^{\leq 2}\Psi) = \mathbf{P}[\neg true \vee succ] \cdot \underline{Prob}^D(\Phi\ U^{\leq 1}\ \Psi) = [\mathbf{0.98}, \mathbf{0.9898}, \mathbf{0}, \mathbf{1}]$$

$\Phi\ U^{\leq k}\ \Psi$ – FOR $k \neq \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{Prob}^D(\Phi\ U^{\leq 2}\ \Psi) = [\mathbf{0.98}, \mathbf{0.9898}, \mathbf{0}, \mathbf{1}]$$

$$\text{Hence } Sat(P_{>0.98}[F^{\leq 2} succ]) = \{s_1, s_3\}$$

$\mathbf{\Phi\ U^{\leq k}\ \Psi}$ – FOR $k = \infty$

- For brevity, instead of $U^{\leq\infty}$ we just write U
- We need to compute $Prob^D(s, \Phi\ U\ \Psi)$ for each state s:

$$Prob^D(s, \Phi\ U\ \Psi) == \begin{cases} 1 & if\ s \in Sat(\Psi) \\ 0 & if\ s \in Sat(\neg\Phi \wedge \neg\Psi) \\ \sum_{s' \in S} \mathbf{P}(s, s') \cdot Prob^D(s', \Phi\ U\ \Psi) & \text{otherwise} \end{cases}$$

$\Phi\ U^{\leq k}\ \Psi$ – FOR $k = \infty$

This system of equations can have many solutions – we convert it to one with just one solution

The following sets are computed using fixpoint algorithm (similar to CTL case, using complement on sets):

$$Sat(P_{\leq 0}[\Phi\ U\ \Psi]) = \{s \in S \mid Prob^D(s, \Phi\ U\ \Psi) = 0\}$$
$$Sat(P_{\geq 1}[\Phi\ U\ \Psi]) = \{s \in S \mid Prob^D(s, \Phi\ U\ \Psi) = 1\}$$

$\Phi\ U^{\leq k}\ \Psi$ – FOR $k = \infty$

Resulting system of equation then reads:

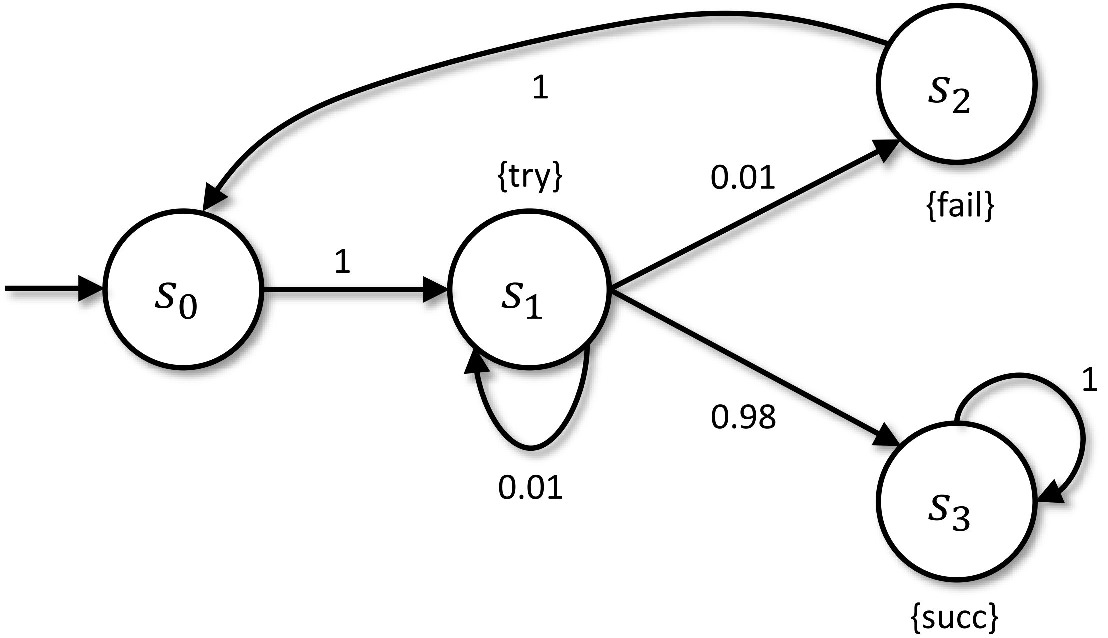
$$Prob^D(s, \Phi\ U\ \Psi) =$$

$$= \begin{cases} 1 & if\ s \in Sat(\mathrm{P}_{\geq 1}[\Phi\ U\ \Psi]) \\ 0 & if\ s \in Sat(P_{\leq 0}[\Phi\ U\ \Psi]) \\ \sum_{s' \in S} \mathbf{P}(s, s') \cdot Prob^D(s', \Phi\ U\ \Psi) & \text{otherwise} \end{cases}$$

Having computed sets for probabilities 0 and 1, we can restrict computation to rest of states

- Optimization

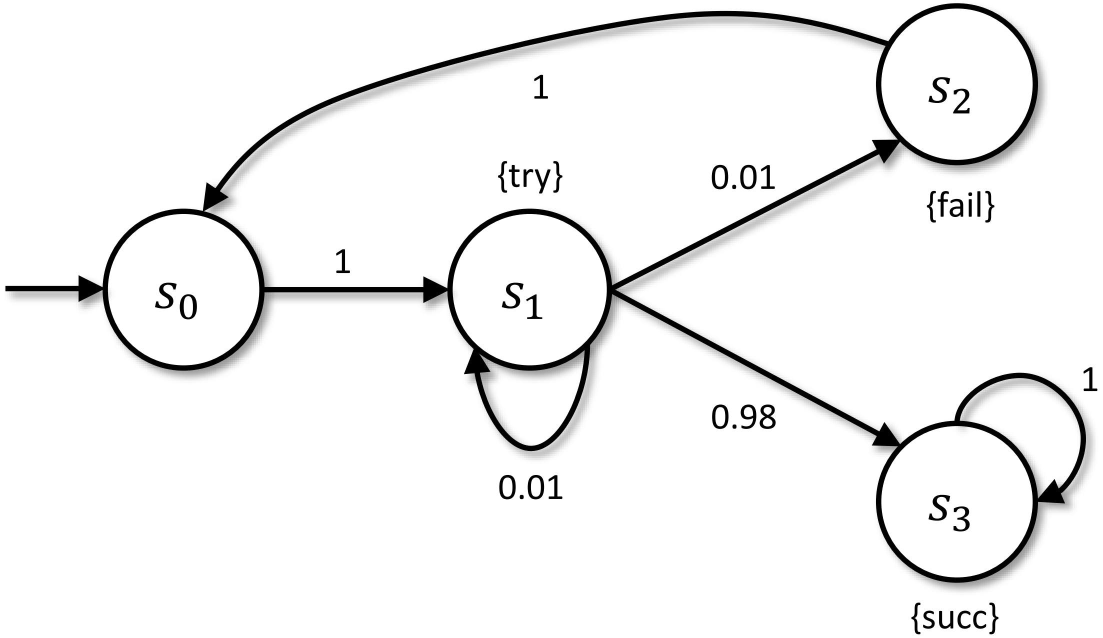
$\Phi\ \mathbf{U}^{\leq k}\ \Psi$ – FOR $k = \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_{>0.99}[try\ U\ succ]$$

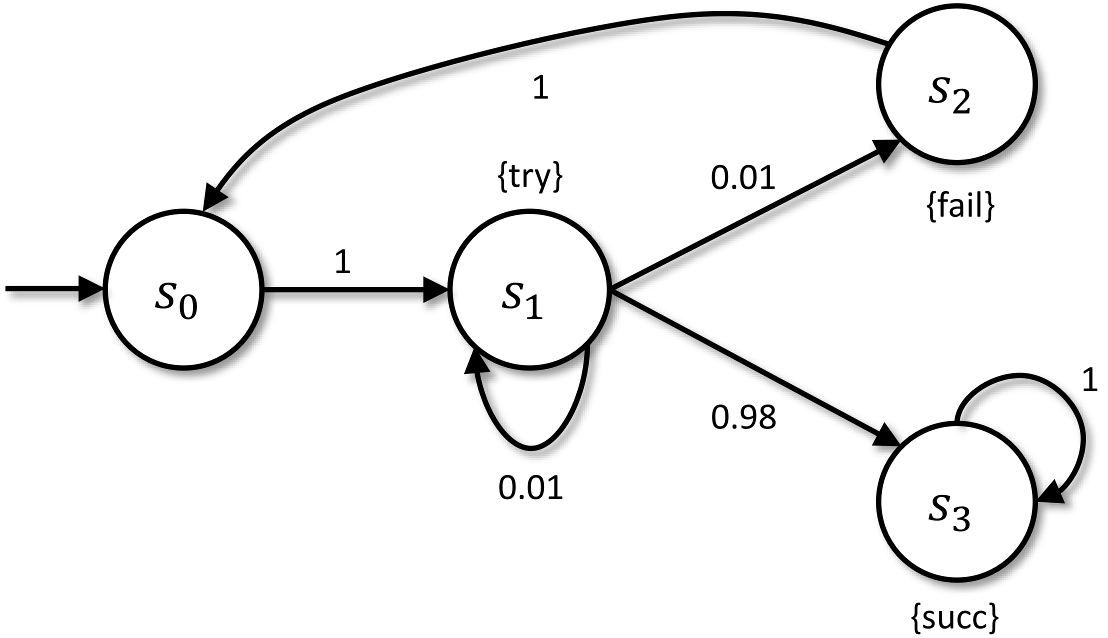
$\Phi\ \mathbf{U}^{\leq k}\ \Psi$ – FOR $k = \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Sat(try) = \{s_1\}, \qquad Sat(succ) = \{s_3\}$$

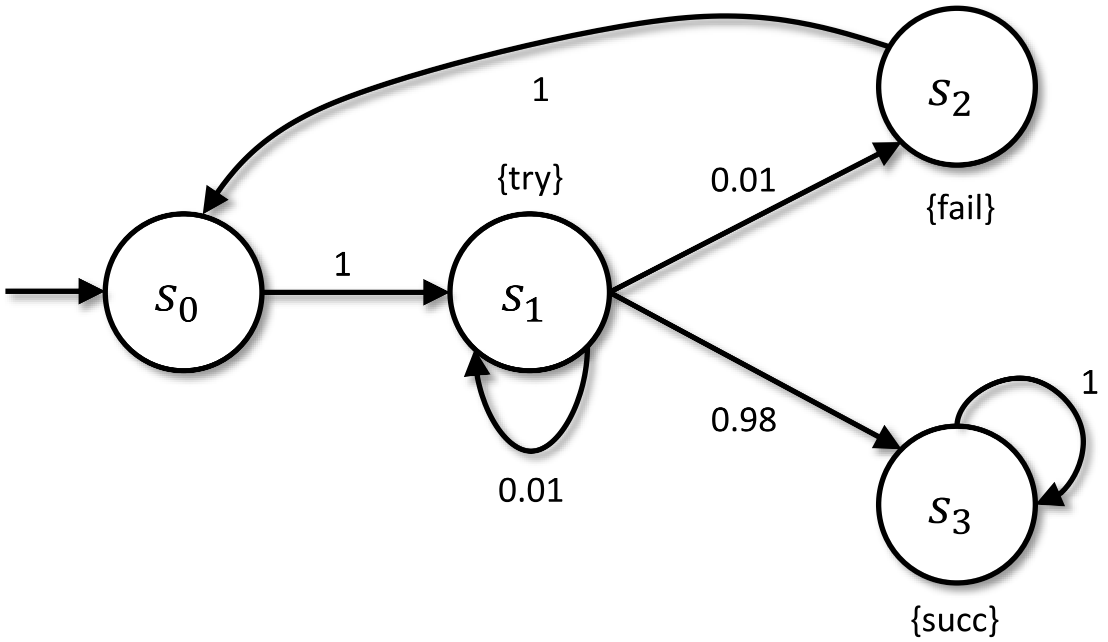
$\Phi\ U^{\leq k}\ \Psi$ – FOR $k = \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Sat(P_{\leq 0}[try\ U\ succ]) = \{s_0, s_2\}, \qquad Sat(P_{\geq 1}[try\ U\ succ]) = \{s_3\}$$

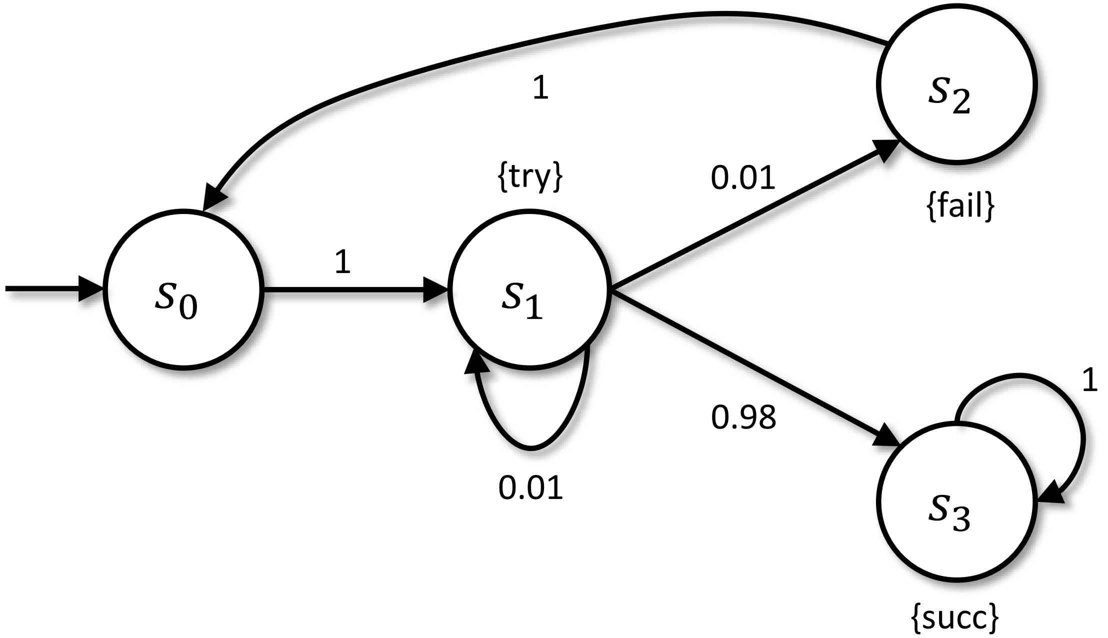
$\Phi\ \mathrm{U}^{\leq k}\ \Psi$ – FOR $k = \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Prob^D(s_0, try\ U\ succ) = 0$$
$$\begin{aligned} Prob^D(s_1, try\ U\ succ) = {} & 0.01 \cdot Prob^D(s_1, try\ U\ succ) + \\ & 0.01 \cdot Prob^D(s_2, try\ U\ succ) + \\ & 0.98 \cdot Prob^D(s_3, try\ U\ succ) \end{aligned}$$
$$Prob^D(s_2, try\ U\ succ) = 0$$
$$Prob^D(s_3, try\ U\ succ) = 1$$

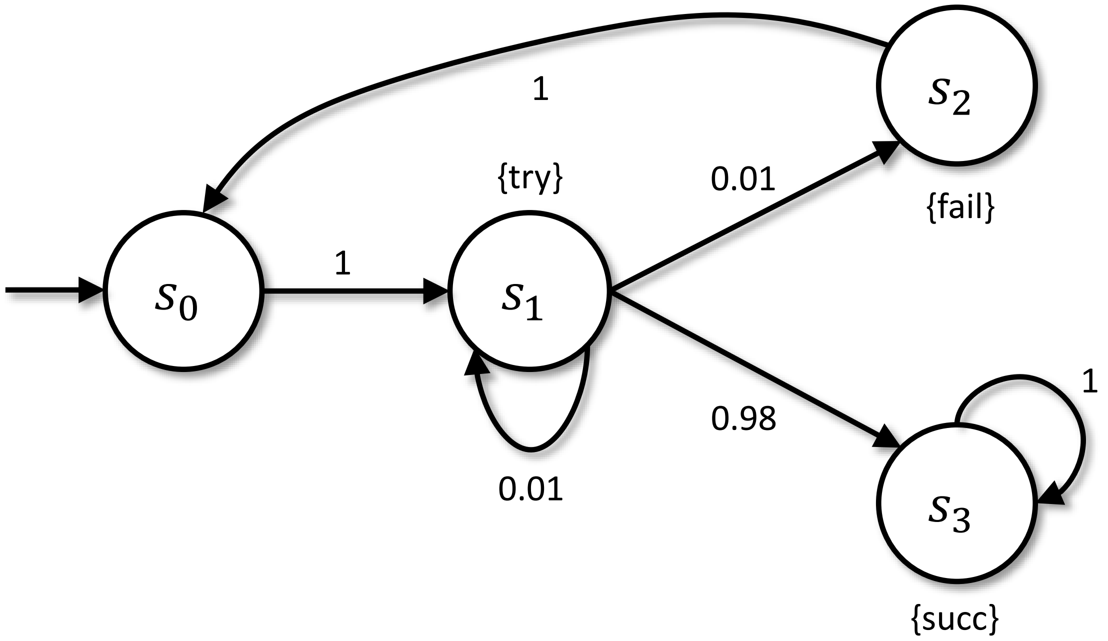
$\Phi\ \mathbf{U}^{\leq k}\ \Psi$ – for $k = \infty$ – Example



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{Prob}^{D}(try\ U\ succ) = \left(0, \frac{98}{99}, 0, 1\right)$$

$\mathbf{\Phi}\ \mathbf{U}^{\leq \mathbf{k}}\ \mathbf{\Psi}$ – FOR $k = \infty$ – EXAMPLE



$$\mathbf{P} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0.01 & 0.01 & 0.98 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{Prob}^D(try\ U\ succ) = (0, \frac{98}{99}, 0, 1)$$

$P_{>0.99}[try\ U\ succ]$ is satisfied in s_3

Extending DTMC and PCTL with rewards

DTMC and PCTL can be extended by rewards (or costs)

- specification of cost for transition
- reasoning about cost of particular computation, e.g., satisfying PCTL property, restricting to computations with cost less than k, …

Continuous-time Markov Chains

CONTINUOUS-TIME MARKOV CHAINS

Transitions are supposed to occur at real time

- contrary to DTMC where they occur at discrete time steps

CTMC allow to reason about different properties

- Continuous Stochastic Logic (CSL) is used instead of PCTL
- very close to PCTL including time specifications
- support for specification of time intervals

Continuous-time Markov Chains

Instead of probability matrix of DTMC, we have **transition rate matrix** (of real numbers)

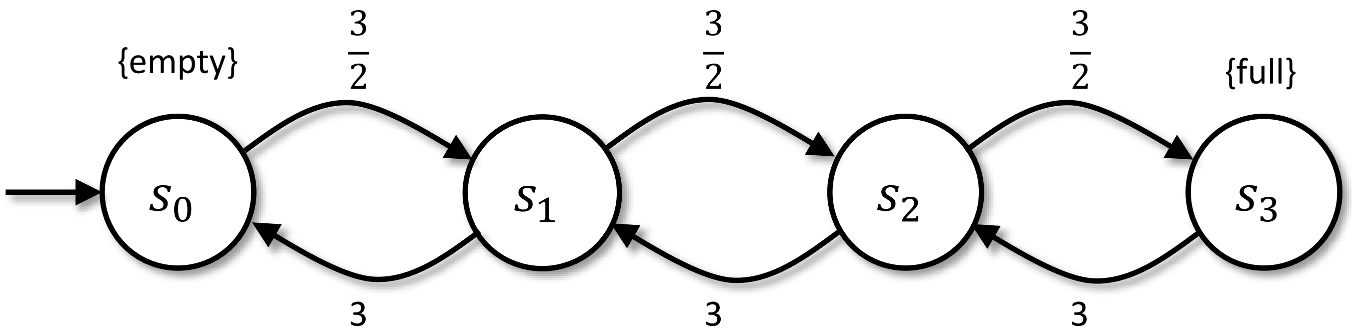
- assigns rates to each pair of states
- rates determine the probability of the transition
- exponential distribution – probability of transition (s, s') within t time units, if $\mathbf{R}(s, s') > 0$ equals

$$1 - e^{-\mathbf{R}(s,s') \cdot t}$$

Exit rate $E(s)$ of state s is given by:

$$E(s) \stackrel{\text{def}}{=} \sum_{s' \in S} \mathbf{R}(s, s')$$

CTMC – Example



$$\mathbf{R} = \begin{pmatrix} 0 & \frac{3}{2} & 0 & 0 \\ 3 & 0 & \frac{3}{2} & 0 \\ 0 & 3 & 0 & \frac{3}{2} \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\mathbf{P}^{\mathbf{emb}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

PRISM – Probabilistic Model Checker

- Allows for checking DTMC, CTMC and other types of models
- Uses simple dedicated input language
- http://www.prismmodelchecker.org

```
// Two process mutual exclusion

mdp

module M1

    x : [0..2] init 0;

    [] x=0 -> 0.8:(x'=0) + 0.2:(x'=1);
    [] x=1 & y!=2 -> (x'=2);
    [] x=2 -> 0.5:(x'=2) + 0.5:(x'=0);

endmodule

module M2

    y : [0..2] init 0;

    [] y=0 -> 0.8:(y'=0) + 0.2:(y'=1);
    [] y=1 & x!=2 -> (y'=2);
    [] y=2 -> 0.5:(y'=2) + 0.5:(y'=0);

endmodule
```