Rewriting Systems

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Motivation: executable specifications

- Systematic rewriting based on equations

- Example: list length

- Rewrite systems
  - Theory (background)
  - Practice (Maude)
Substitution

- Signature: the set $V$ of variable names

- Substitution $\sigma : V \rightarrow X$
  - Unifier of $t_1$ and $t_2$ if $\sigma(t_1) = \sigma(t_2)$

- Inductively defined sub-expressions

- Reducible sub-expression $t_1[\beta] \, // \, \text{redex}$
  - If $\sigma(\beta) = \sigma(t_2)$
Rewriting rules & systems

- Rule \( r : l \rightarrow p \)
- Application of a rule
  - \( \sigma(t)[\beta \leftarrow \sigma(p)] \)
- Rewriting system: set \( R \) of rules
- Derivation \( t \rightarrow^R u \)
  - Reflexive transitive closure \( \rightarrow^R \)
- Irreducible expressions
  - Normal form (canonical)
Properties of rewriting systems

- Confluence
  \[ \forall t, t_1, t_2 \ ( ((t \xrightarrow{R^*} t_1 \land t \xrightarrow{R^*} t_2) \Rightarrow \exists u \ (t_1 \xrightarrow{R^*} u \land t_2 \xrightarrow{R^*} u)) \]

- Terminating
  - Normal form always exists

- Canonical
  - Single normal form
Canonical values

- Classes of equivalent terms (expressions)
  - Generated by equations (sentences) in the set E

- Canonical representatives of classes

- Canonical values (forms) of expressions
Knuth-Bendix procedure

- **Input:** $Q = (S, \Sigma, E), \leq \subseteq X \times X$

- **Algorithm**
  1) $R := \emptyset$
  2) if $E == \emptyset$ then return $R$ // canonical rewriting system
  3) take any $t_1 = t_2 \in E$ such that either $t_1 \leq t_2$ or $t_2 \leq t_1$
     - 3a) if $\exists t_1 = t_2$ then $E := E - \{t_1 = t_2\}$
     - 3b) if $t_1$ and $t_2$ not comparable then fail // $R$ cannot be created
  4) if $t_2 \leq t_1$ then $R := R \cup \{R(t_1) \rightarrow R(t_2)\}$
  5) if $t_1 \leq t_2$ then $R := R \cup \{R(t_2) \rightarrow R(t_1)\}$
  6) if $R(t_1) \neq R(t_2)$ then $E := E \cup \{R(t_1) = R(t_2)\}$
  7) continue with step 3
Connection to algebraic specifications

- Equations
  - Simple rewriting semantics (simplification)
  - Left-hand side replaced by right-hand side
Maude

- Web: [http://maude.cs.illinois.edu/](http://maude.cs.illinois.edu/)
  - source code, documentation, examples
- Version: 2.7.1

- Main features
  - Functional modules and theories
    - Algebraic specifications
  - Numeric and string data types
  - Computation (rewriting, equations)
    - membership equational logic
  - much more (check the web site)
Maude: installation & running

- **Linux**

- **Windows**
  - [https://www.cygwin.com/](https://www.cygwin.com/)

- **Running**
  - `<directory with Maude>\maude.exe`
  - From the command-line in a working directory that contains your input files
Maude: basic commands

• 1) Prepare specification in a text file

• 2) run the Maude tool

• 3) load your input file: load <file>

• 4) apply rewriting on some expression
   reduce [in <module> :] <expr>

• 5) Exit the Maude prompt: quit
Maude programs: syntax and semantics

- Functional modules
  - sorts, variables, operations, equations

- Notation for operations: prefix, mixfix

- Comments

- Built-in sorts and modules
  - Bool, NAT, INT, FLOAT, RAT, QID, STRING
Examples
- Natural numbers (Peano arithmetic)
- Stack of natural numbers

Theories

Conditional equations

Membership axioms

Attributes of operations
Maude programs: advanced concepts

- Parameterized modules (generic)
  - Example: generic stack

- Importing modules
  - protecting
  - extending
  - including
Maude programs: there is even more

- Data structures (MAP, ARRAY, others)
- Rewriting rules ("basic", conditional)
- Useful built-in modules (CONFIGURATION)
Maude and Rewriting Logic

- [http://maude.cs.illinois.edu/papers/](http://maude.cs.illinois.edu/papers/)