Making IP = PSPACE Practical: Efficient Interactive Protocols for BDD Algorithms

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collaboration with

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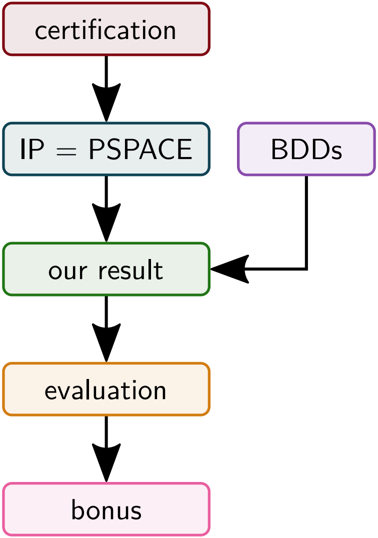
September 11, 2023

Outline



Is this formula satisfiable?

$$
\begin{aligned}
& (x \lor y \lor \neg z) \\
\land\ & (\neg x \lor \neg z \lor w) \\
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No... at least my SAT-solver says so!

Certification

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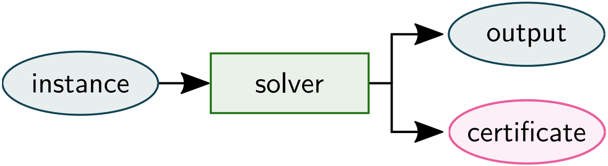
- Use certification – each answer comes with a machine-checkable certificate



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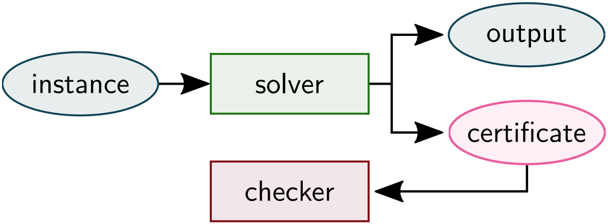
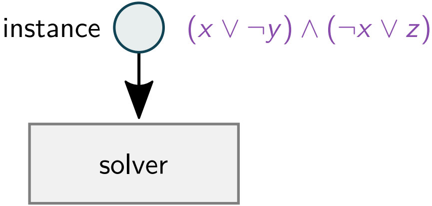


- It suffices to ensure correctness of the certificate checker

SAT – boolean satisfiability



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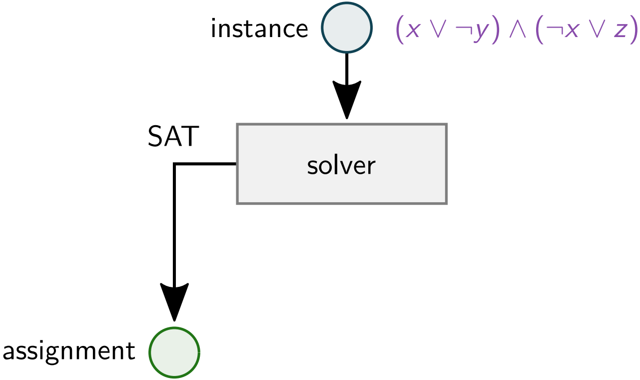

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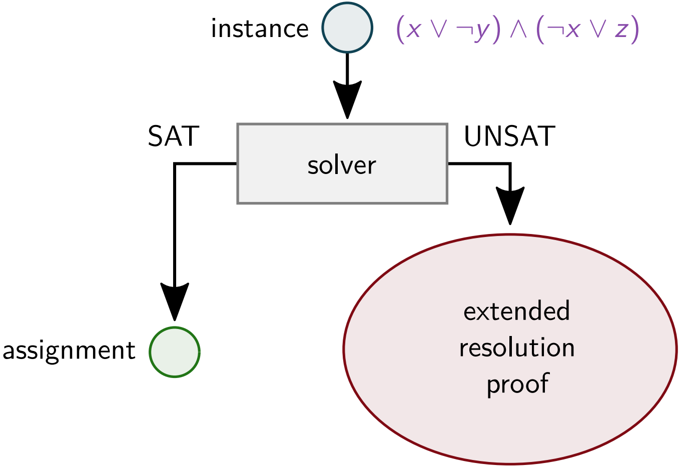
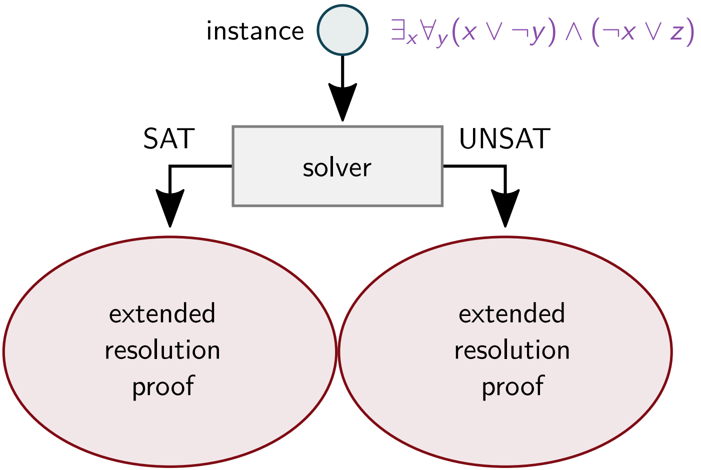


QBF – quantified boolean satisfiability



This talk applies to QBF as well.

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- Used for (UN)SAT, QBF

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- Essentially a list of clauses, each of which is implied by the previous clauses
- Properties:
  - "efficiently" checkable
  - long (exponential in size of the input)
- Certificates can be many terabytes (!) in size
  - e.g. 200 TiB in [Heule,Kullmann,Marek 2016] to solve the boolean Pythagorean Triples problem

The problem

- Huge resolution proofs are difficult to handle

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- In some cases, it can take even longer to verify the proof than to solve the instance (!)

Polynomial-time certification?!

No.

No. However...

Interactive Protocols – Summary

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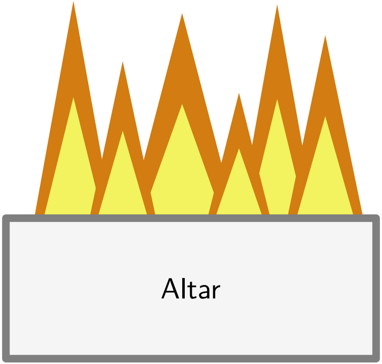

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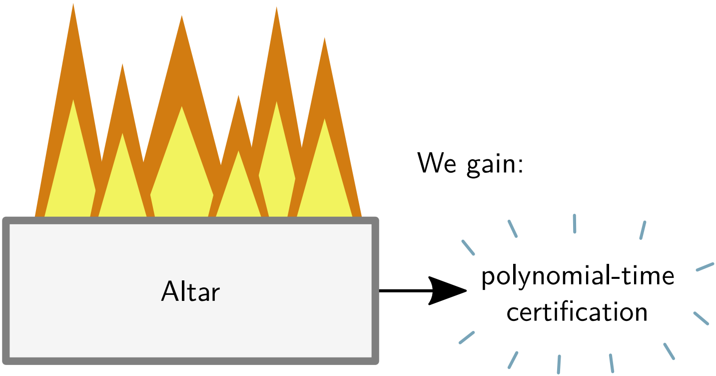

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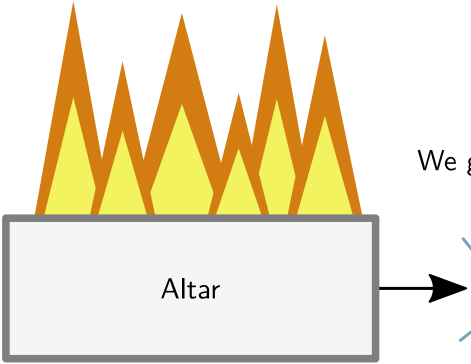

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A breakthrough

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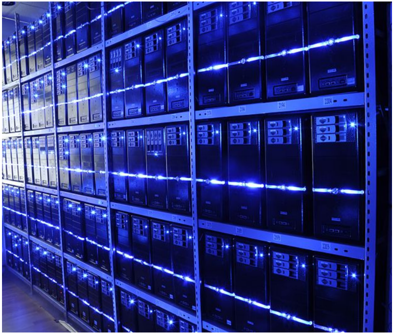
- demonstrates that efficient certification is possible via interactive protocols, for *any* PSPACE problem
  - i.e. SAT, QBF, model counting, ...

Interactive Protocols

Verifier

Prover



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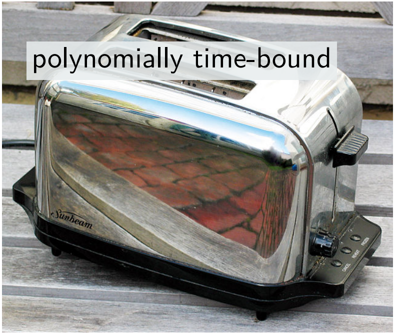

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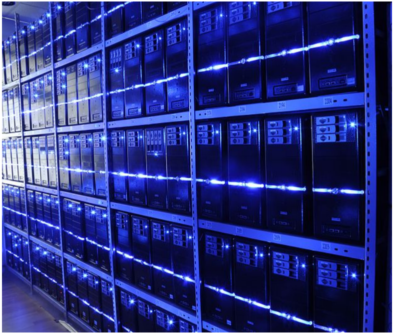

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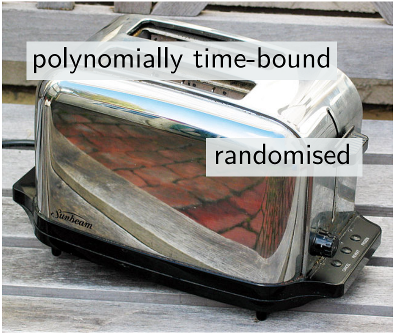

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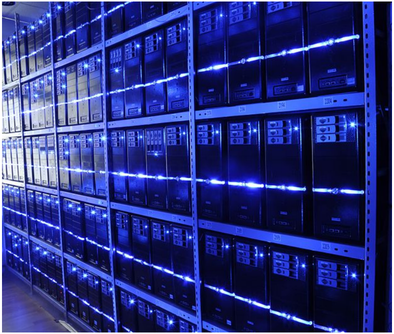

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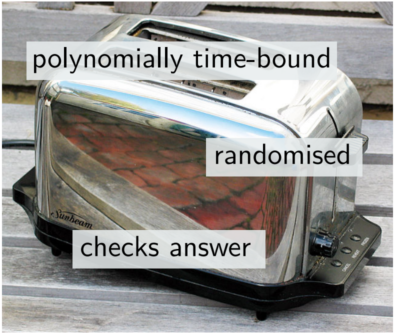

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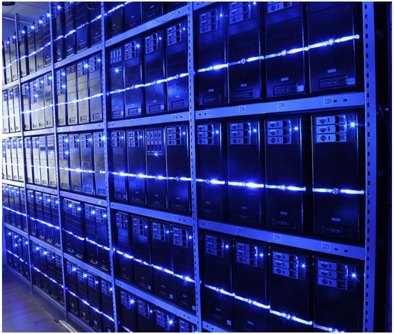

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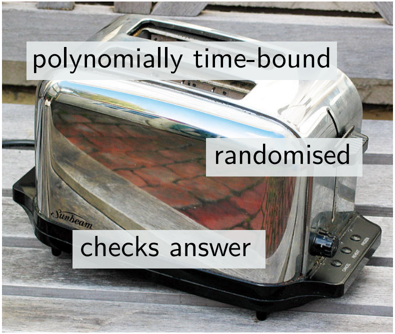

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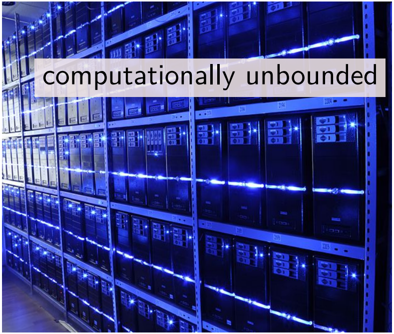

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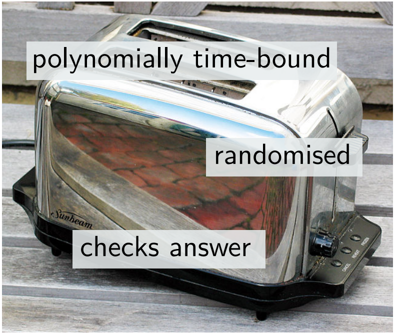

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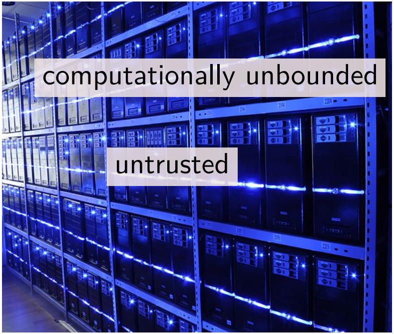

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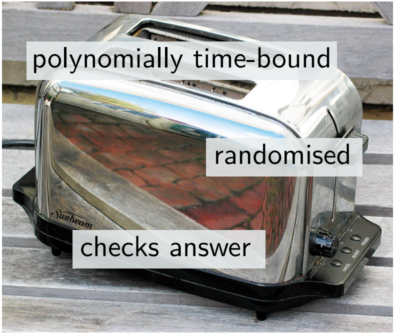

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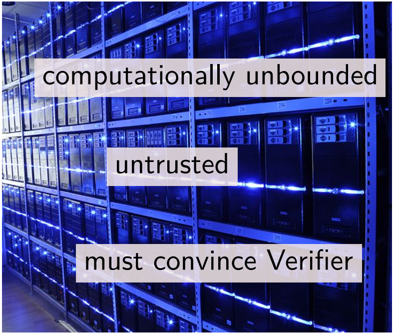

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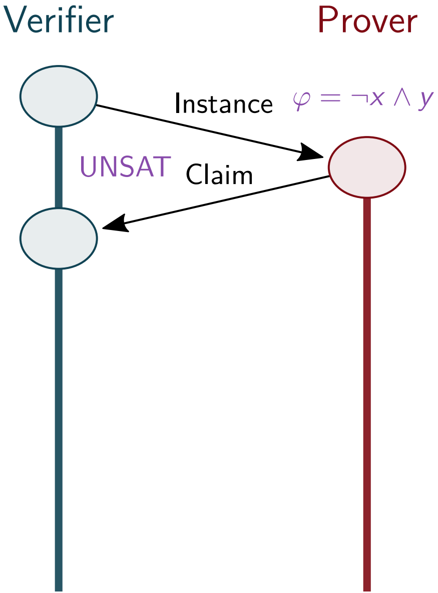

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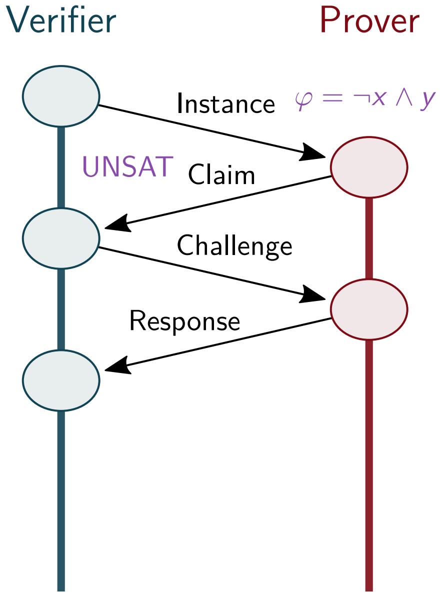

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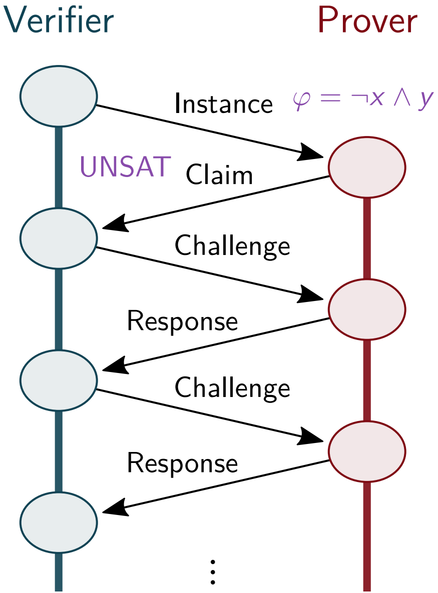

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- IP is the class of problems that admit such a protocol

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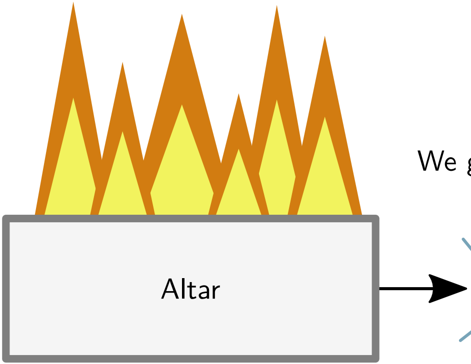

We gain:

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Problem: how do we generate interactive certificates with practical approaches?

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- Reduced Ordered Binary Decision Diagrams (BDDs)

$$x \wedge (y \oplus z)$$
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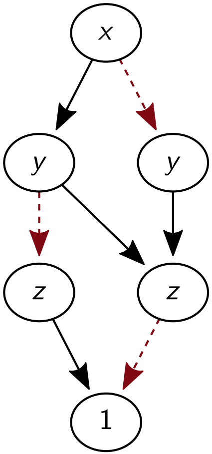

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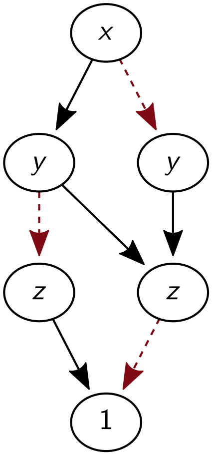

BDDs

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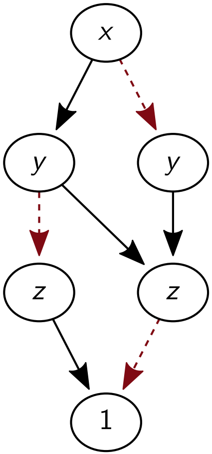
- Reduced Ordered Binary Decision Diagrams (BDDs)
- Unique encoding of boolean functions with efficient boolean operations
- Are used effectively for QBF, CTL model checking (and many other problems)
  - not as good for SAT, though



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Theorem. Let φ denote a QBF instance with n variables.

1. Verifier executes in time $\mathcal{O}(n^2|\varphi|) \approx 0$, with negligible failure probability $\approx 10^{-10}$, and
2. Prover takes $\mathcal{O}(T) \approx 3T$ time to solve φ and answer Verifier's challenges,

where T is the time the BDD algorithm takes to solve φ.

(constants in practice)

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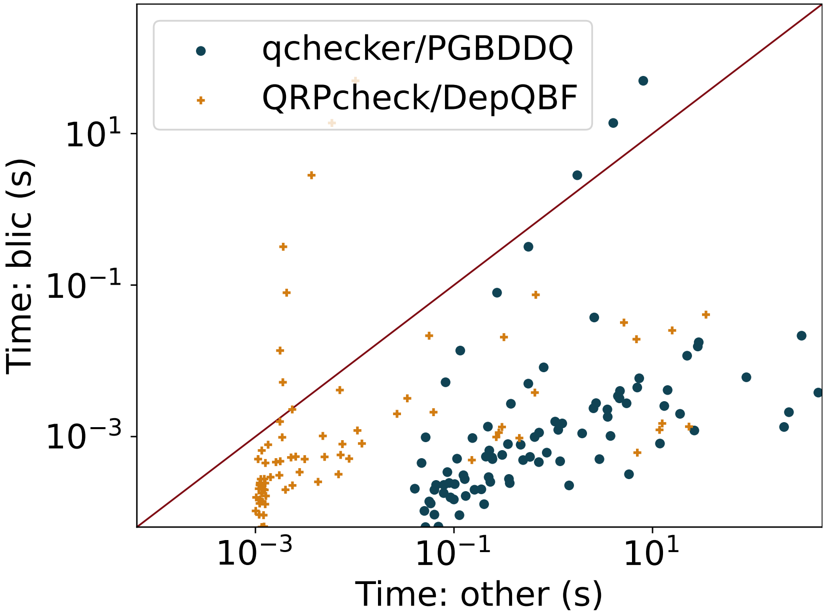
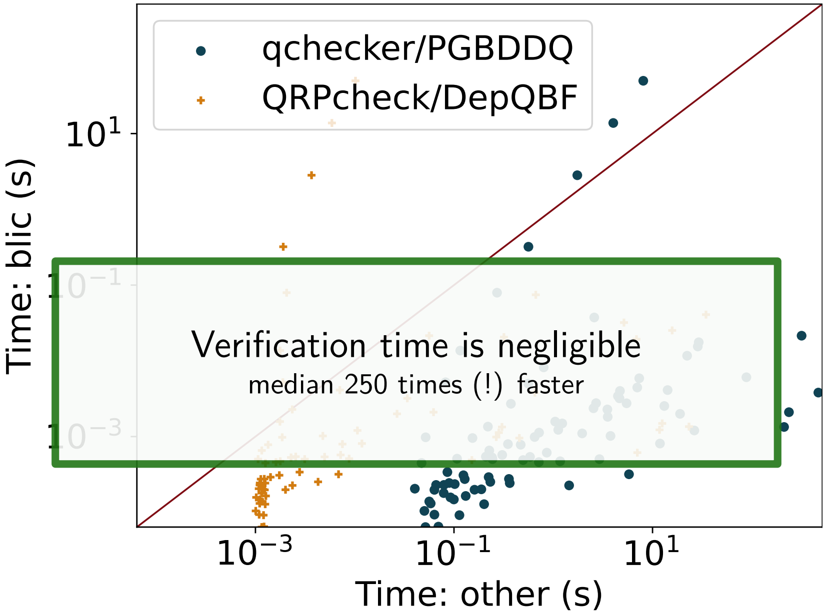
- We compare against state-of-the-art QBF solvers CAQE, DepQBF and PGBDDQ
- DepQBF and PGBDDQ are certifying as well, using extended resolution proofs
- Benchmarks are taken from the crafted instances track of the QBF Evaluation 2022



Time to verify certificate (Verifier / external specialised checkers)



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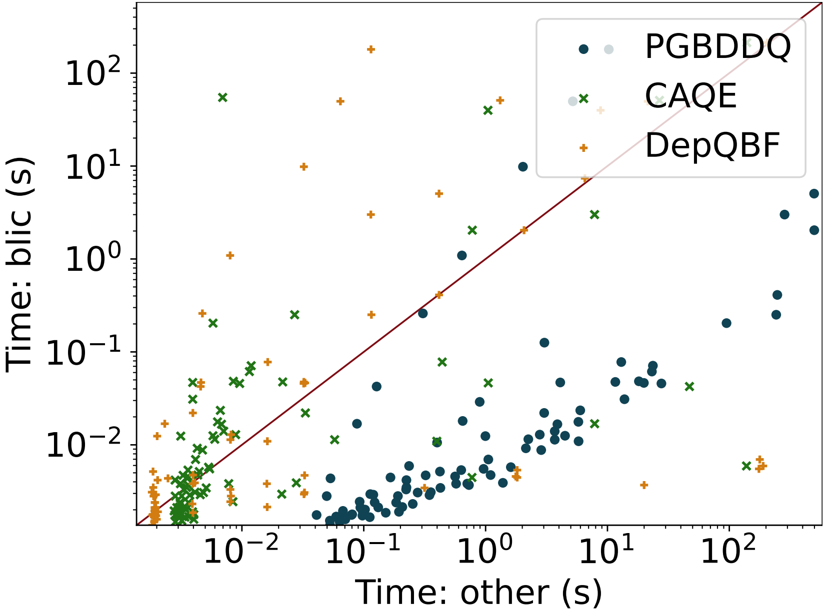

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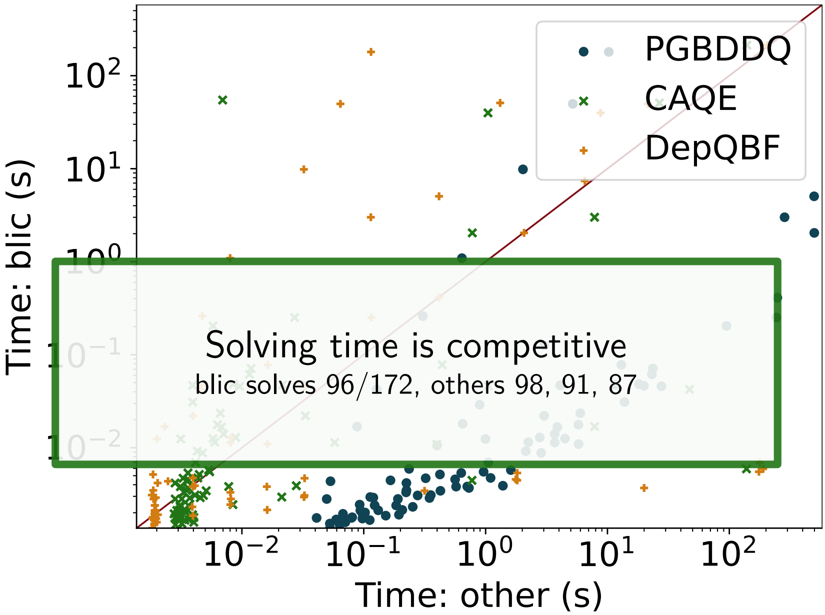

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- Competitive performance (blic solves 96 of 172 benchmarks, others 98, 91 and 87)
- Generating interactive certificates is low-overhead (factor $\sim$3)
- Error probability is negligible (at most 10^{-10} here)
- Can be applied to any BDD algorithm

More Power!

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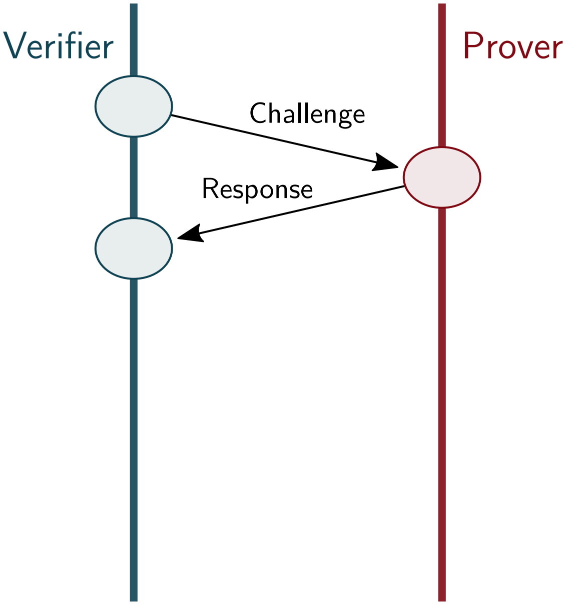
Using this one simple trick...

Resets



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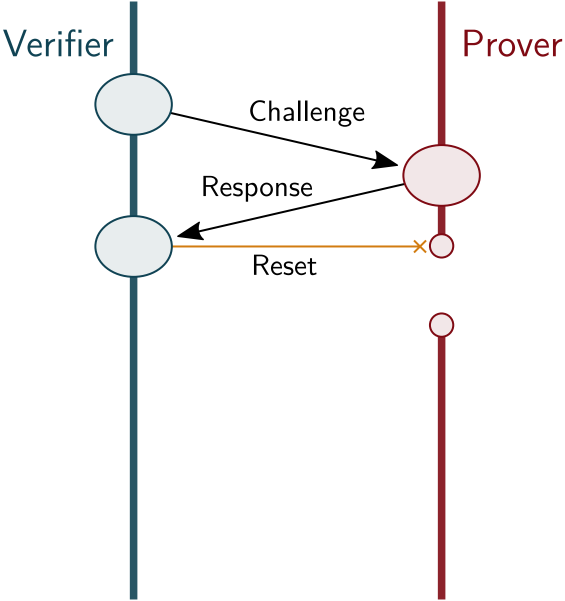

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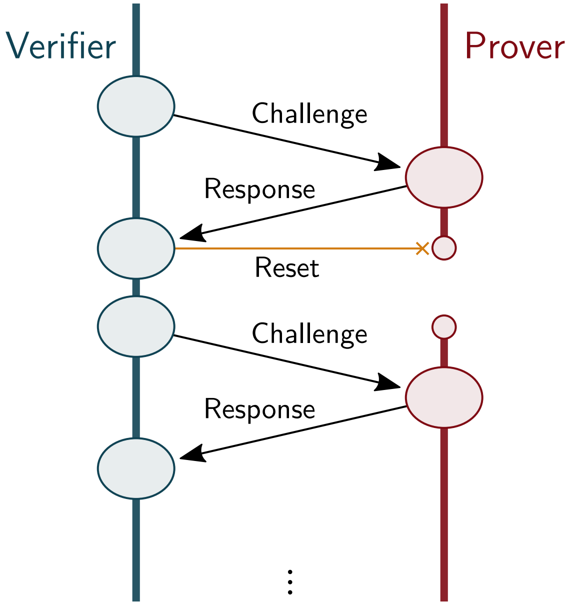
Verifier
Prover
Challenge
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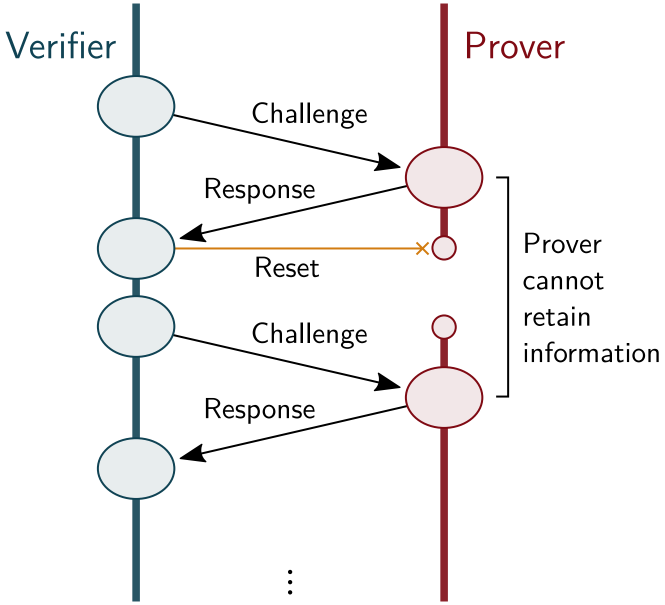
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Prover cannot retain information

Resets

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- This increases the power to NEXP
- Seems reasonable in practice

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Use-case 1: Avoid memory overhead

- Current approach needs to persist intermediate BDDs to disk
- Essentially, computation is bottom-up, while certification is top-down
- With a reset-button, we can run the certification on-the-fly

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  - Prover needs poly($|\Phi|$) time
- With a reset-button, this seems possible

Conclusions

First practical approach with polynomial-time certificate verification!

- Checking time of the interactive certificate are negligible (median 250 times faster!)
- Competitive performance (blic solves 96 of 172 benchmarks, others 98, 91 and 87)
- Generating interactive certificates is low-overhead (factor ~ 3)
- Error probability is negligible ($\leq 10^{-10}$)
- Can be applied to any BDD algorithm

Thank you for your attention! Questions?