

What is a Function?

The story of a concept, from Leibniz to 1900

David Corfield

Two pictures of a function

A function is a rule

```
def square(x):  
    return x * x
```

- You can *follow* it
- It is finite: you can write it down

A function is a table (a graph)

x	0	1	2	3	...
$f(x)$	0	1	4	9	...

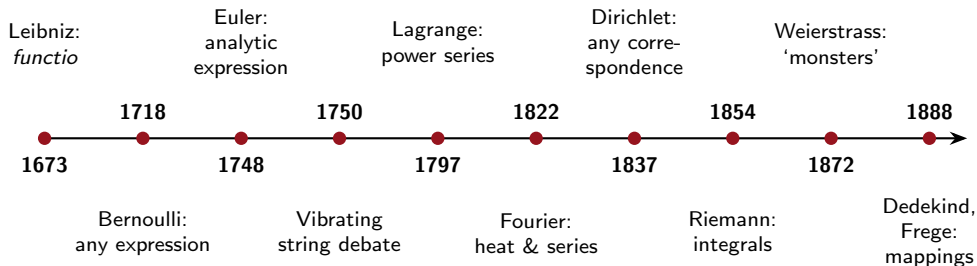
- Only the input–output pairs matter
- No rule need exist

Is $x*x$ the same function as $x*x + 1 - 1$?

Different rules, same table — so which picture decides?

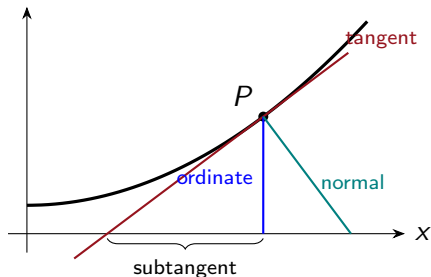
Mathematics took about 200 years to move from the first picture to the second.
Computing calls that move into question.

The timeline



Three successive answers: a function is **a geometric quantity** → **a formula** → **an arbitrary correspondence**.

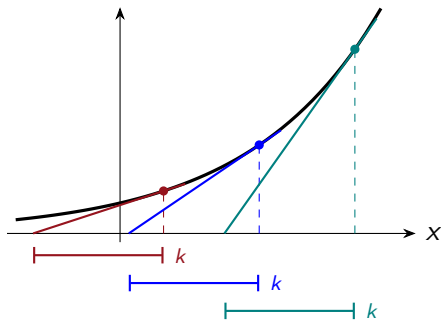
Leibniz (1670s–1690s): functions live on curves



- Leibniz called segments like these *functiones*, from Latin *fungi*, “to perform a duty”
- Each has a particular **job** relative to the curve at P : touching it, measuring its height, standing perpendicular
- As P moves along the curve, their lengths change: the first hint of **a quantity depending on another**

The curve still comes first. No formulas yet, and no idea of input and output.

An inverse tangent problem



- The *inverse* problem: we know something about the tangent at every point; find the curve
- Example: the subtangent always has the **same length** k
- The tangent drops the height y over a distance k , so slope = height / k
- A curve that grows in proportion to its own height: the **exponential** $y = Ce^{x/k}$

The “function” (subtangent) is given; the curve is the unknown.

Bernoulli (1718) and Euler (1748): a function is a formula

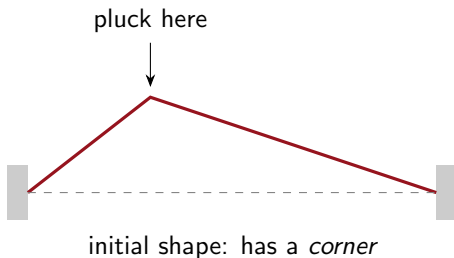
- **Johann Bernoulli**, roughly: a function of a variable is a quantity *composed in any manner* from that variable and constants
- **Euler**, *Introductio* (1748), roughly: a function is an **analytic expression** (*expressio analytica*) built from the variable, numbers and constants

$$x^2 + 1, \quad \frac{1}{1-x}, \quad \sin x + e^x, \quad 1 + x + x^2 + x^3 + \dots$$

- The function is now the main object of study, not the curve
- Euler split functions into two kinds:
 - “**continuous**”: one formula all the way along
 - “**mixed**” / “**discontinuous**”: different formulas on different pieces, or a curve drawn freely by hand

Warning: two old words. “Analytic” here means *symbolic/algebraic* (as in “analysis” vs. geometry), not the modern “analytic function”. And Euler’s “continuous” is *not* our “continuous”.

A physics problem forces the issue: the vibrating string



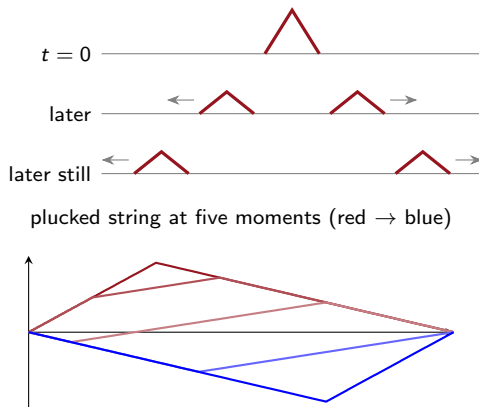
- D'Alembert (1747) wrote down the general motion of a string
- The answer involves functions describing the string's starting shape
- **Question:** which starting shapes are allowed?
- A plucked string is *not* one smooth formula

Is “a shape you can make with your fingers” a function?

D'Alembert's solution: two travelling copies

- The starting shape splits into **two half-height copies**
- One moves left, one moves right, at the same speed
- At a fixed end, a copy bounces back upside down
- Corners never smooth out: they just travel

D'Alembert's worry: the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ uses the rate of change of the slope. At a corner the slope jumps, so the equation makes no sense there.

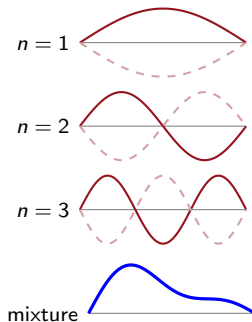


Daniel Bernoulli: harmonics

- A **mode** is a sine shape that simply swings up and down (solid $\leftrightarrow$ dashed)
- Mode n has n humps and swings n times as fast: the musical **harmonics**
- Bernoulli: every motion of the string is a *mixture* of modes

$$y = b_1 \sin \pi x + b_2 \sin 2\pi x + \cdots$$

- Can a mixture of smooth sines ever make the plucked string's corner?



The debate (1747–1760s)

d'Alembert	Only proper formulas (smooth ones) are allowed. The plucked string lies outside the mathematics.
Euler	Physics needs arbitrary shapes, including corners and freehand curves. So we must allow “mixed” functions.
Daniel Bernoulli	Every motion is a sum of simple sine waves (the harmonics).
Euler & d'Alembert	Impossible! A sum of sines is <i>one</i> formula, so it cannot equal a shape made of different pieces.

- Hidden assumption: **one formula = one “law” = one kind of behaviour everywhere**
- In 1755 Euler also floated a much looser definition: *if one quantity changes when another changes, it is a function of it*. It did little work for decades.

Lagrange (1797): every function is a power series

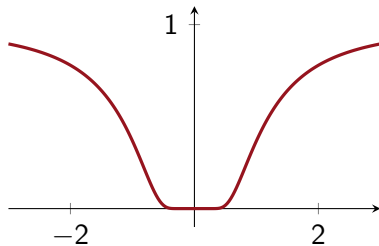
- Goal: do calculus without mysterious “infinitely small” quantities
- Assume every function can be written

$$f(x + h) = f(x) + a_1 h + a_2 h^2 + \dots$$

- Then *define* the derivative as a_1
- Formula view at its most ambitious

Cauchy (1820s): the function on the right has all derivatives 0 at $x = 0$, so its series is $0 + 0h + 0h^2 + \dots$, which is wrong.

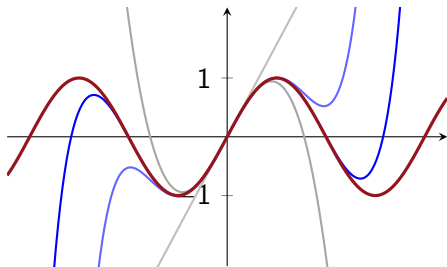
$$f(x) = e^{-1/x^2}, f(0) = 0$$



Very flat at 0 — but not zero.

When power series work — and when they don't

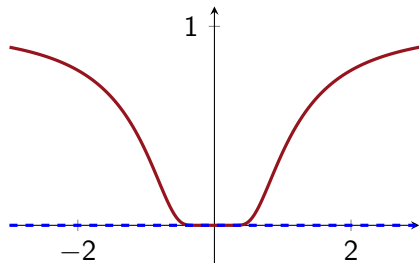
$\sin x$ and its polynomial approximations



More terms, longer agreement:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

e^{-1/x^2} and *all* its approximations at 0



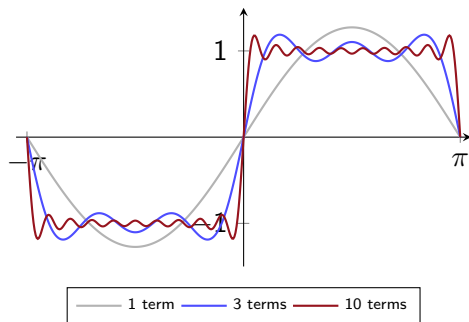
Every polynomial approximation at 0 is the flat line $y = 0$. Lagrange's programme cannot see this function.

Fourier (1807–1822): heat and arbitrary functions

- How does heat spread through a metal bar?
- Fourier's bold claim: **any** function on an interval is a sum of sines and cosines
- He gave formulas for the coefficients
- Leading mathematicians (incl. Lagrange) had doubts about rigour

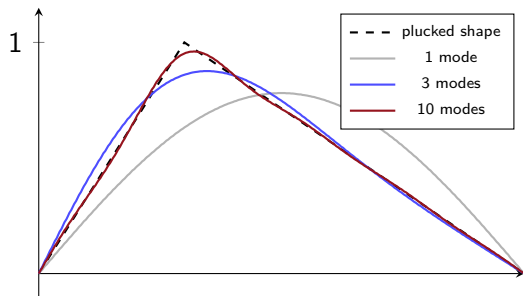
Even a function with jumps can be built from smooth waves.

Building a square wave



Bernoulli vindicated: sines can make a corner

The plucked string as a sum of modes



- The coefficients of the modes can be calculated (Fourier's formulas)
- Here they shrink like $1/n^2$, so the sum settles down fast
- With ten modes the corner is already sharp
- Euler's "mixed" function *is* a single sum of sines

The square wave's coefficients shrink only like $1/n$: slower, and jumps are harder than corners.

Why Fourier broke the formula picture

- The square wave is $+1$ on one piece and -1 on the other: a “mixed” function in Euler’s sense
- Yet it equals **one single formula**, an infinite sum of sines
- So “given by one formula” does *not* separate functions into two kinds

Even simpler (Cauchy, 1844)

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} \quad \text{but also} \quad |x| = \sqrt{x^2}$$

Two formulas or one? It depends on how you write it down.

Lesson: **the same function can have many descriptions.**

(Computing echo: many programs, one input–output behaviour.)

Cauchy (1821): continuity, and a famous mistake

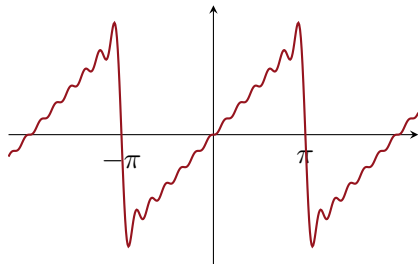
- Cauchy gave our modern idea of **continuity**: small change in input $\Rightarrow$ small change in output
- He also “proved”: an infinite sum of continuous functions is continuous
- Abel (1826): look at a Fourier series!

$$\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots$$

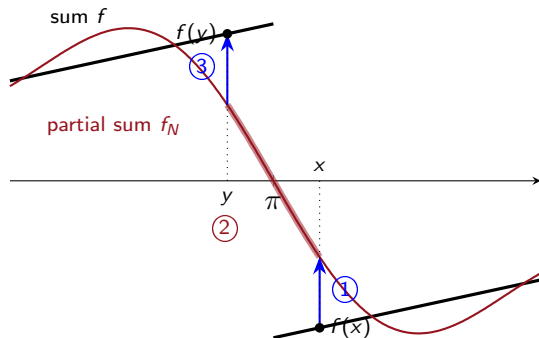
Each term is continuous; the sum
jumps

The fix (“uniform convergence”) took until
about 1850.

12 terms: a sawtooth appears



Where Cauchy's proof breaks



To show the sum f is continuous, compare $f(x)$ with $f(y)$ for nearby x, y by a detour through a partial sum f_N :

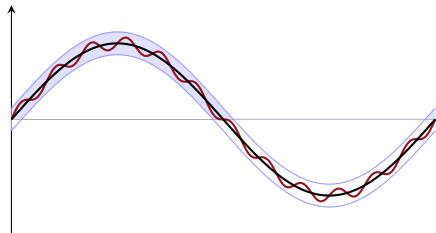
- ① from $f(x)$ to $f_N(x)$
- ② along the curve f_N from x to y
- ③ from $f_N(y)$ back to $f(y)$

② is small: f_N is a finite sum of continuous functions.

①, ③ shrink if N is large enough — but near the jump, “large enough” keeps growing. More terms just make f_N climb more steeply, closer to π .

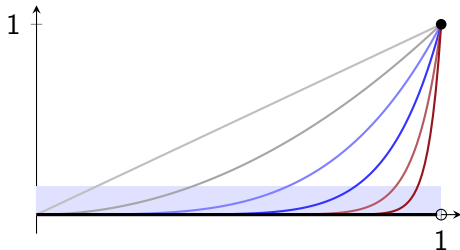
The repair: uniform convergence

Uniform: the whole graph fits in the band



$\sin x + \frac{\sin 12x}{12}$ stays within $\frac{1}{12}$ of $\sin x$ everywhere

Not uniform: x^n always escapes near 1



Limit: 0 for $x < 1$, but 1 at $x = 1$ — a jump

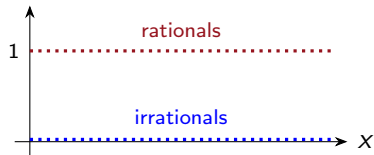
- **Pointwise:** at each x separately, $f_N(x)$ eventually gets close
- **Uniform:** one N puts the *whole graph* of f_N inside the band. Then ① and ③ are small at every point, and the limit **is** continuous

Dirichlet (1829, 1837): any correspondence at all

- First real proof that Fourier series work, under clear conditions (finitely many jumps and wiggles)
- 1837, roughly: y is a function of x if each x gets **one** y . No common law, no formula needed.
- His strange example:

$$D(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$$

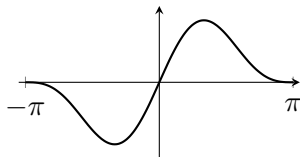
Jumps *everywhere*; impossible to draw



Historians' caution: Dirichlet himself mostly worked with well-behaved functions. The "Dirichlet definition" label was largely attached later.

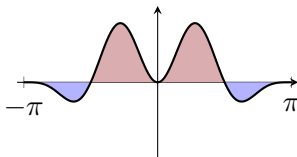
How Fourier finds the coefficients: multiply and take the area

$$f(x) = \sin x + \frac{1}{2} \sin 2x$$



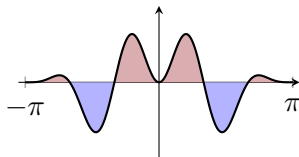
A function made of two waves.
How do we recover “how much
 $\sin 2x$ ” it contains?

$$f(x) \cdot \sin 2x$$



More **above** than **below**: the
area gives $b_2 = \frac{1}{2}$

$$f(x) \cdot \sin 3x$$



Above and **below** cancel
exactly: $b_3 = 0$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (\text{the area under } f(x) \sin nx, \text{ rescaled})$$

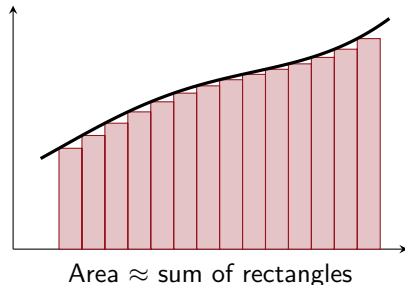
Multiplying by $\sin nx$ and taking the area **filters out** every wave except the n th.

No area, no coefficients: so **which functions have an area?**

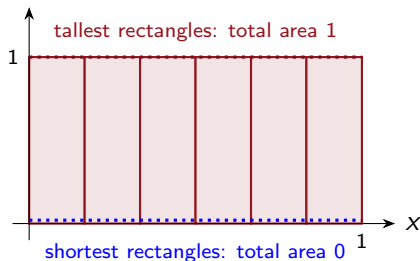
Changing f at a single point changes no area — so two different functions (say, square waves with value 0 or 1 at the jump) share one Fourier series.

Riemann (1854): when can we find the area?

- To find how much of each sine wave is in f , Fourier multiplies f by that sine and takes the area. So: **which functions have an area?**
- Riemann's answer: chop into thin strips, add rectangles, see if it settles down
- He built a function with *infinitely many* jumps, packed densely, that still has a well-defined area
- Bad behaviour no longer disqualifies a function



Why Dirichlet's function has no (Riemann) area

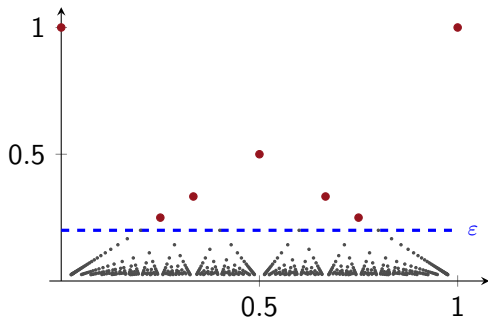


- Riemann: chop $[0, 1]$ into strips; in each strip use the highest value, then the lowest
- Every strip, however thin, contains rationals (value 1) *and* irrationals (value 0)
- So the “highest” total is always 1 and the “lowest” always 0
- Thinner strips never bring them together: **no area**

A perfectly well-defined function that the best integral of the day cannot handle.

Thomae's function: dense jumps, yet an area

$$T(p/q) = 1/q, \quad T(\text{irrational}) = 0$$



The “popcorn” function (denominators up to 40)

- Same split as Dirichlet's function, but rationals with big denominators sit **low**
- **Continuous at every irrational:** rationals close to it must have big denominators, so nearby values are near 0
- **Jumps at every rational:** irrationals with value 0 are arbitrarily close
- **Area 0:** above any height ε only *finitely many* dots stick up (red). Cover them with very thin strips; everywhere else the tallest value is below ε

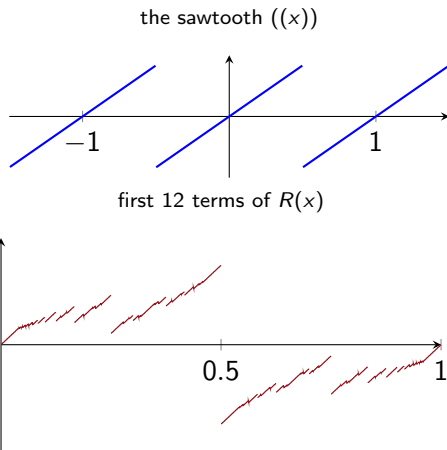
Compare: for Dirichlet's function the tallest rectangles always total 1.

Riemann's example: densely many jumps, but still an area

- Start with a sawtooth $((x))$: x minus the nearest whole number (jumps at every half-integer)
- Riemann added squeezed, shrunken copies:

$$R(x) = ((x)) + \frac{1}{4}((2x)) + \dots$$

- The sum jumps at every fraction with an even denominator: in every interval, however small
- But the jumps get small so fast that the strips still settle down:
the area exists



Weierstrass (1872): the “monsters”

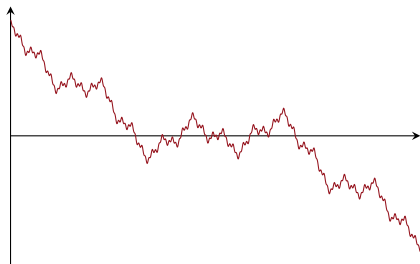
- A function that is **continuous everywhere** (no jumps) but has **no slope anywhere** (corners at every scale)

$$W(x) = \sum_{n \geq 0} a^n \cos(b^n \pi x)$$

- Intuition said: a continuous curve must be smooth somewhere
- Reactions: Hermite turned away “in horror”; Poincaré called such functions inventions made only to show earlier proofs were faulty

Are these legitimate objects, or pathologies?

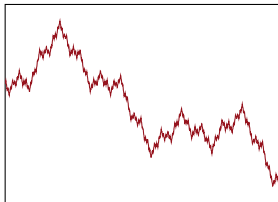
$a = \frac{1}{2}$, $b = 3$, six terms



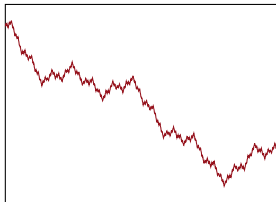
Zoom in: it never straightens out

Zooming in on Weierstrass's function

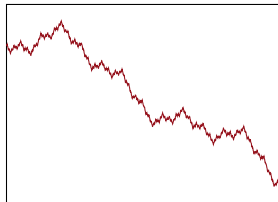
window width 1



zoom $\times 9$



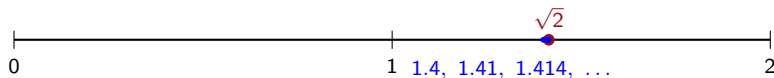
zoom $\times 81$



- Zoom in on a smooth curve and it looks more and more like a **straight line**: that line's slope is the derivative
- Zoom in on W and it stays just as rough, at every scale and at every point
- So there is no slope anywhere — yet the curve never jumps

Meanwhile: what is a real number?

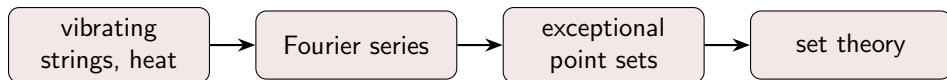
Geometry	The line is simply continuous; a variable “flows” along it (Euler)
The gap	Proving that a curve which goes from below the axis to above it really crosses it needs a property of numbers , not a picture (Bolzano 1817, Cauchy 1821)
Construction	Real numbers built from rationals, by cuts or by sequences (Dedekind, Cantor, both 1872). Cantor built his <i>for</i> his Fourier series work
Structure	“Nearness” becomes limit points and ε -distances: the seeds of topology



A real number is now an *infinite* object. So what can a program do with one?

Cantor (1870s): from Fourier series to set theory

- Question: if two *trigonometric* series (sums of sines and cosines) converge to the same values everywhere, must they have the same coefficients?
- Cantor (1870): **yes** — and he proved it without assuming uniform convergence, which made it much harder
- What if they agree everywhere *except* at some bad points?
- Cantor needed to describe which sets of exceptional points are harmless
- This led him to study **infinite sets of points** for their own sake, and then to different sizes of infinity



Set theory later supplies the official definition of “function”.

Dedekind (1888) and Frege (1891): functions in general

Dedekind: mappings

- A mapping sends each element of one collection to an element of another
- Inputs need not be numbers at all
- Treated as a basic *logical* idea

Frege: functions vs. expressions

- “ $x^2 + 1$ ” is an *expression*; the function is what it denotes
- A function is “unsaturated”: it has a gap waiting for an argument

$$(\cdot)^2 + 1$$

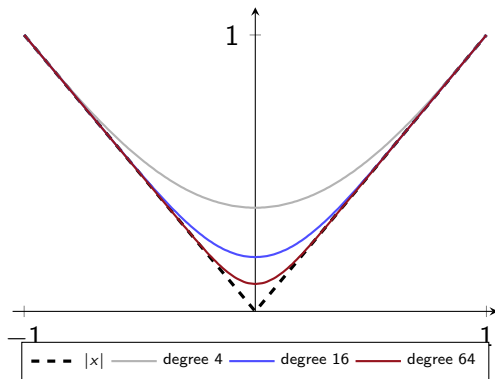
- A concept is a function to *True/False*:

$$\text{“} \cdot \text{ is prime”} : 7 \mapsto \text{True}$$

Compare: `lambda x: x**2 + 1, is_prime : int -> bool`

Closing a loop: every continuous function is a limit of polynomials

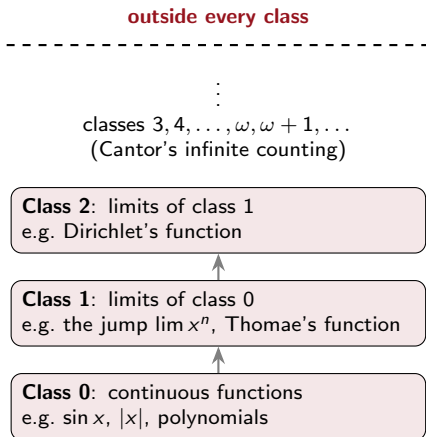
Polynomials closing in on $|x|$



- Euler asked, in effect: is every function given by a formula?
- Each curve is a single polynomial: smooth, no corner
- Raising the degree squeezes them into any band around $|x|$: the convergence is **uniform**
- Weierstrass (1885): this works for *every* continuous function on an interval
- Euler's "one formula" returns — but only as a *limit*

The recipe shown is Bernstein's.

Beyond continuous functions: limits of limits



- **Baire (1899)**: start from continuous functions and ask how many **rounds of taking limits** a function needs
- Even Dirichlet's "lawless" function needs only two — it has a formula, of a sort:

$$D(x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (\cos(m! \pi x))^{2n}$$

Rational $x = p/q$: once $m \geq q$, $m!x$ is whole, so the cosine is ± 1 . Irrational x : never whole, and powers of a number below 1 shrink to 0.

- **Lebesgue (1905)**: every rung is occupied, and the whole ladder is exactly the functions expressible by formulas and limits
- But he also showed that **some functions lie above the ladder**

Cantor's diagonal trick (1891): most functions have no formula

input	0	1	2	3	4	5	
f_0	0	0	1	1	0	0	...
f_1	1	1	0	0	1	1	...
f_2	1	0	0	0	1	0	...
f_3	0	0	0	1	0	1	...
f_4	1	1	1	1	1	0	...
$\vdots$							
g	1	0	1	0	0	...	

- Try to **list** all functions from $\{0, 1, 2, \dots\}$ to $\{0, 1\}$: $f_0, f_1, f_2, \dots$ (each row is one function's table)
- Walk down the diagonal and flip each bit: $g(n) = 1 - f_n(n)$
- g differs from f_n at input n , so g is *nowhere* in the list
- So no list can contain all such functions
- But formulas and programs are *finite strings*, so they *can* be listed (length 1, then length 2, ...)
- So **most functions have no formula and no program**

Summary: three pictures of a function

Picture	Main figures	What pushed it
Geometric quantity	Leibniz	curves, tangents
Formula / expression	Bernoulli, Euler, Lagrange	algebra, calculus
Arbitrary correspondence	Fourier, Dirichlet, Riemann, Dedekind	strings, heat, series, integrals

- Physics demanded more functions than formulas seemed to give
- Infinite processes (series, limits) blurred the line between “formula” and “not”
- The general definition admitted “monsters”, and not everyone liked them

Looking ahead: the twentieth century and computing

- **Set theory (1910s)**: a function is a set of input–output pairs, each input appearing once. The “table” picture becomes official.
- **Constructive mathematics** (Brouwer from the 1900s; later Markov, Bishop): a function must come with a **construction**. You cannot in general decide whether a real number is rational, so Dirichlet’s function cannot be defined on all reals — and for Brouwer, every function on the reals turns out to be continuous
- **1930s**: Church (λ -calculus), Gödel, Turing ask: which functions can be computed by a **mechanical rule**?

The “rule” picture returns, now made precise

- Recall the diagonal trick: programs can be listed, functions cannot — so **most functions have no program**. Which ones *do*?
- **Extension vs. intension**: when are two programs “the same function”?

Questions to take away

1. Is the Dirichlet function a function in the same sense as square? What does a programmer gain or lose by the general definition?
2. Euler thought one formula meant one law everywhere. Write a one-line program, with no `if`, that follows one rule on some inputs and a different rule on others: say, output 0 when x is negative but x itself when x is positive.
3. Should a definition of “function” rule out the monsters? Who gets to decide?
4. Frege’s “unsaturated” function and `lambda x: ...`: same idea or different?
5. Is this a story of **progress**? Which intuitions — a function as a curve you can draw, or a law you can follow — were lost when “function” came to mean “any correspondence”? Do computing or constructive mathematics win any back?

Further reading

- I. Kleiner, “Evolution of the function concept: a brief survey”, *College Mathematics Journal* 20 (1989). Short and accessible.
- A. P. Youschkevitch, “The concept of function up to the middle of the 19th century”, *Archive for History of Exact Sciences* 16 (1976).
- U. Bottazzini, *The Higher Calculus: A History of Real and Complex Analysis from Euler to Weierstrass* (1986).
- I. Grattan-Guinness, *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (1970).
- I. Lakatos, *Proofs and Refutations* (1976), appendix on Cauchy’s sum theorem.
- G. Frege, “Function and Concept” (1891), in many translations.