# NSWI101: System Behaviour Models and Verification

# 6. Symbolic CTL Model Checking

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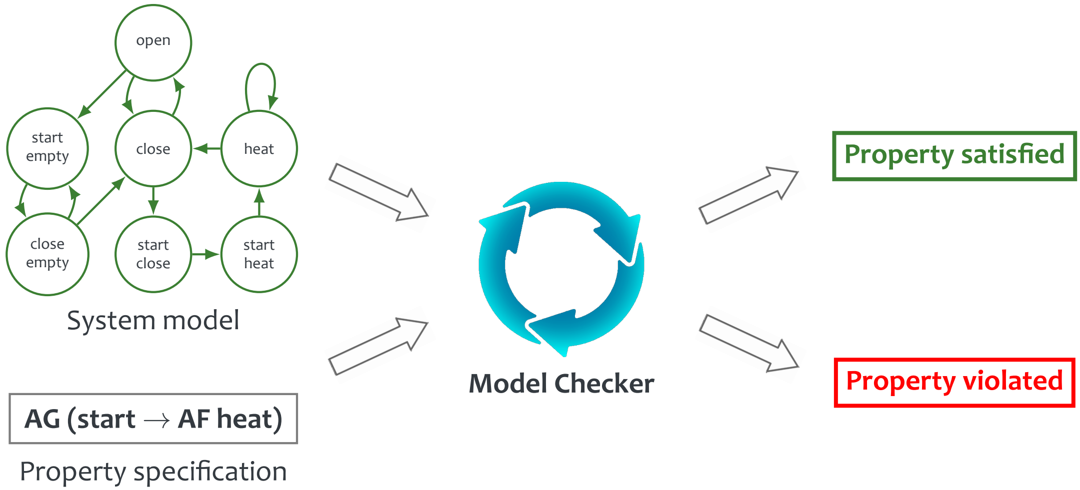
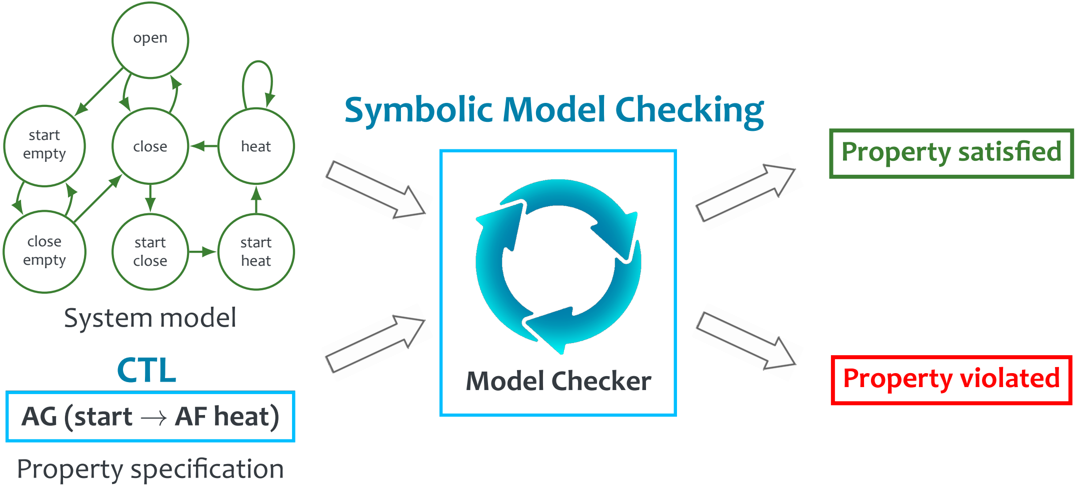
# TODAY

- Symbolic CTL model checking using
  - OBDD
  - lattices
  - fixpoints

# Model Checking



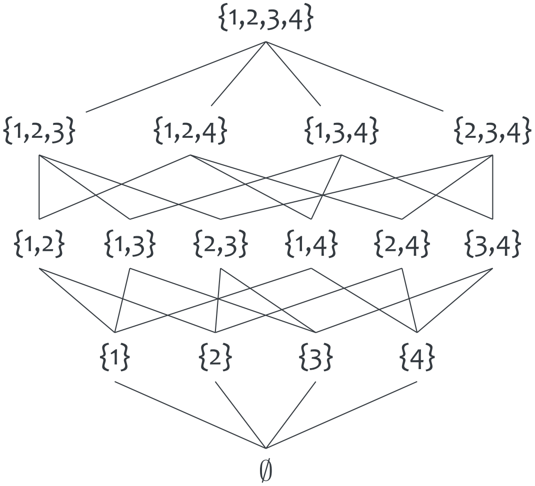
# MODEL CHECKING



# Recall: Lattice

- *Lattice* $L$ is structure consisting of partially ordered set $S$ of elements where every two elements have
  - unique *supremum* (least upper bound or join) and
  - unique *infimum* (greatest lower bound or meet)
- Set $P(S)$ of all subsets of $S$ forms complete lattice
- Each element $E \in L$ can also be thought as *predicate* on $S$
- Greatest element of $L$ is $S$ ($\top$, true)
- Least element of $L$ is $\emptyset$ ($\bot$, false)
- $\tau : P(S) \mapsto P(S)$ is called *predicate transformer*

# Example: Subset Lattice of {1, 2, 3, 4}



# Fixpoints

Let $\tau : P(S) \mapsto P(S)$ be predicate transformer

- $\tau$ is *monotonic* $\equiv Q \subseteq R \implies \tau(Q) \subseteq \tau(R)$
- $Q$ is *fixpoint* of $\tau \equiv \tau(Q) = Q$

# Fixpoint Computation

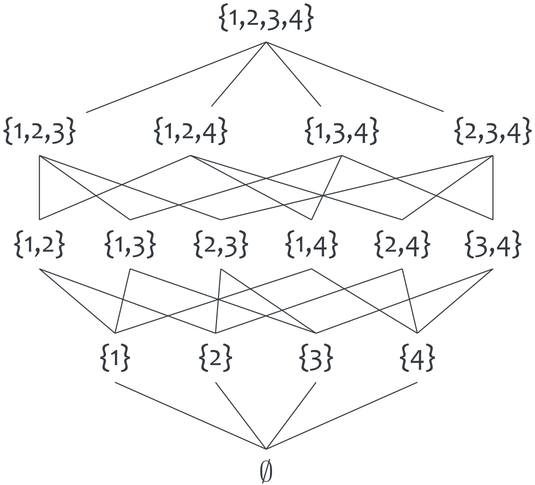
**function** $\textsc{Lfp}(\tau : \mathit{PredicateTransformer})$**:** Predicate
  $Q := \mathit{false}$
  $Q' := \tau(Q)$
  **while** $Q \neq Q'$ **do**
    $Q := Q'$
    $Q' := \tau(Q)$
  **end while**
  $\mathit{return}(Q)$
**end function**

Function Gfp differs just in initialization $Q := \mathit{true}$

# Example of Fixpoints

Let $\tau(Q) = Q \cup \{1\}$.

What are fixpoints of $\tau$?



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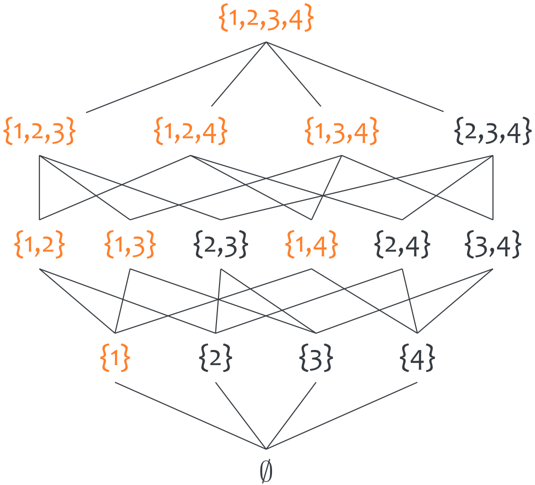

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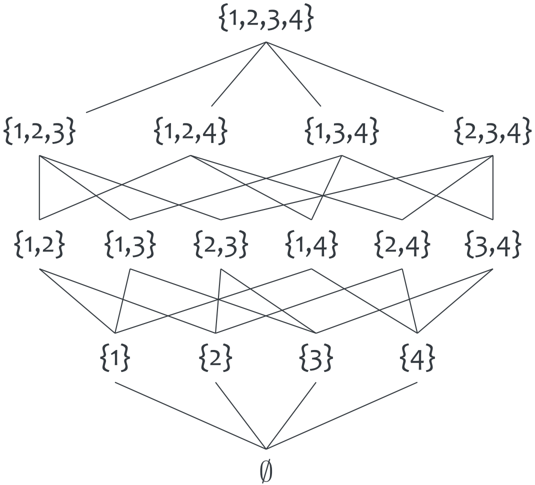

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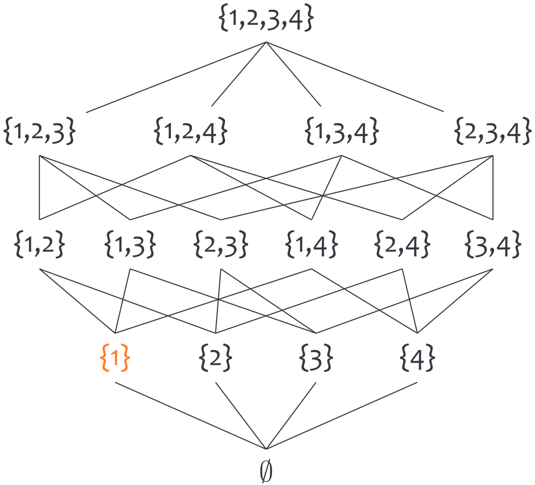
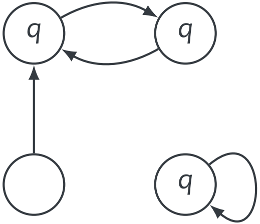


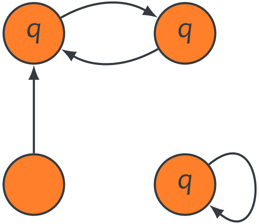
# CTL Operators as Fixpoints

- We identify CTL formula $f$ with set/predicate $\{s | M, s \models f\}$ in $P(S)$
- EG and EU may be characterized as least or greatest fixpoints of an appropriate predicate transformer:
  - $\mathrm{EG}\, q = \nu Z.(q \wedge \mathrm{EX}\, Z)$
  - $\mathrm{E}[p \,\mathrm{U}\, q] = \mu Z.\big(q \vee (p \wedge \mathrm{EX}\, Z)\big)$
- The same holds for EF, AG, AF, AU, however, those operators can be expressed using EG, EU
- Intuitively:
  - least fixpoints correspond to eventualities
  - greatest fixpoints correspond to properties that should hold forever

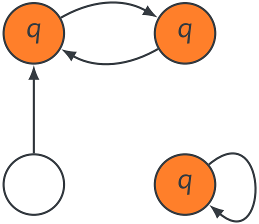
# EG as Fixpoint




Input KS M




$\tau^0(\top)$



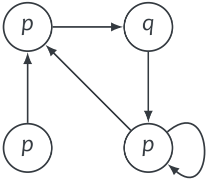

$\tau^1(\top)$

$M, s_0 \models \mathrm{EG}\, q$

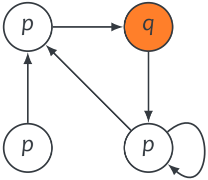
$\mathrm{EG}\, q = \nu Z.(q \wedge \mathrm{EX}\, Z)$

$\tau(Z) = \{s : s \models q \wedge (\exists t : s \to t \wedge t \in Z)\}$

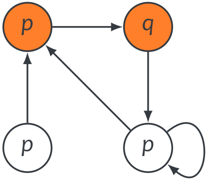
# EU AS FIXPOINT



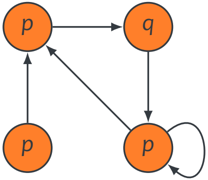

Input KS M




$\tau^1(\perp)$




$\tau^2(\perp)$




$\tau^3(\perp)$

$$M, s_0 \models \mathrm{E}[p \,\mathrm{U}\, q]$$

$$\mathrm{E}[p \,\mathrm{U}\, q] = \mu Z.(q \vee (p \wedge \mathrm{EX}\, Z))$$

$$\tau(Z) = \{s : s \models q\} \cup \{s : s \models p \wedge (\exists t : s \rightarrow t \wedge t \in Z)\}$$

# SYMBOLIC CTL MODEL CHECKING

Explicit model checking—e.g., Spin—is linear in size of generated state space

- usually exponential in size of input model
- resulting in state space explosion

Symbolic model checking operates on sets of states in each step of algorithm

- can mitigate state-space-explosion impact substantially

# QUANTIFIED BOOLEAN FORMULAE

QBFs are useful in symbolic CTL model checking

Quantification does not introduce greater expressive power:

- $\exists x\, f \equiv f|_{x=\bot} \vee f|_{x=\top}$
- $\forall x\, f \equiv f|_{x=\bot} \wedge f|_{x=\top}$

# Symbolic CTL Model Checking

General approach identical to explicit model checking

- decomposing formula into sub-formulae
- identifying sets of states satisfying particular sub-formulae

Computing states satisfying particular formula types based on manipulation with OBDDs

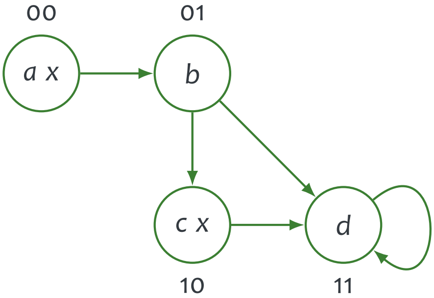
# Symbolic CTL Model Checking

Computing OBDD($f$) for formula $f$ depends on top-most operand

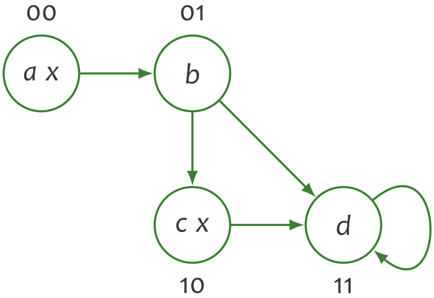
- note that only $\neg$, $\wedge$, $\vee$, EX, EG, and EU are needed, others can be eliminated

- $f \in AP$: return OBDD defined for $f$
- $f : \neg g, f \wedge g$, or $f \vee g$: use logical operation upon OBDD
  - described in previous lecture
- $f = \mathrm{EX}\, g$: OBDD for $\exists \langle v' \rangle \big( o(\langle v' \rangle) \wedge R(\langle v \rangle, \langle v' \rangle) \big)$
  - $o(\langle v \rangle)$ stands for OBDD representing states satisfying formula $g$
- $f = \mathrm{E}[f \,\mathrm{U}\, g]$: compute least fixpoint $\mathrm{E}[f \,\mathrm{U}\, g] = \mu Z. \big( g \vee (f \wedge \mathrm{EX}\, Z) \big)$
  - using LfP procedure
- $f = \mathrm{EG}\, f$: compute greatest fixpoint $\mathrm{EG}\, f = \nu Z. (f \wedge \mathrm{EX}\, Z)$
  - using GfP procedure

# Example



# Example




$$\mathrm{AF}\,x = \neg\mathrm{EG}\,(\neg x)$$

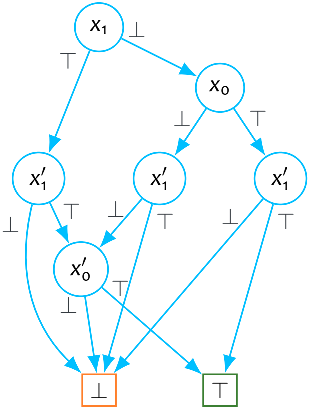
$$\mathrm{EG}\,(\neg x)$$

$$\neg x$$

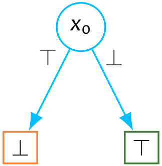
$$x$$

# Example

TR:

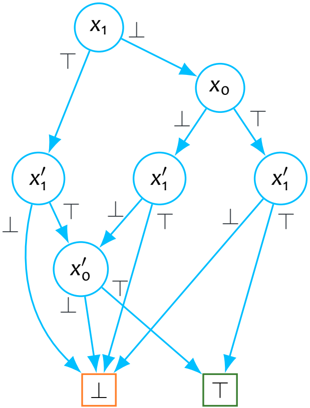



OBDD for states satisfying *x*:

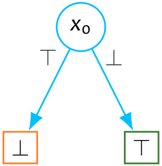


# EXAMPLE

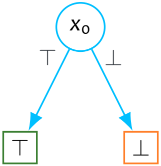
TR:




OBDD for states satisfying $x$:




OBDD for states satisfying $\neg x$:



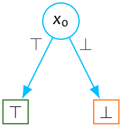
# Example

- We have OBDD for states satisfying $\neg x$ and now, we can proceed to EG $(\neg x)$ and compute OBDD for it.
- We compute *greatest fixpoint* of predicate transformer: EG $(\neg x) : \nu Z.(\neg x \land \text{EX}\, Z)$.
  - computation starts with trivial OBDD for $\top$ $(Z)$.
  - single step: $Z = \neg x \land (\exists x_0', x_1' : Z' \land TR)$
    - $Z'$ denotes OBDD $Z$ where all variables get primed $(x \rightarrow x')$
  - if $Z$ changes, repeat previous step, otherwise fixpoint reached and computation is over

# Example

- We have OBDD for states satisfying $\neg x$ and now, we can proceed to $\mathsf{EG}\,(\neg x)$ and compute OBDD for it.
- We compute *greatest fixpoint* of predicate transformer: $\mathsf{EG}\,(\neg x) : \nu Z.(\neg x \land \mathsf{EX}\, Z)$.
  - computation starts with trivial OBDD for $\top$ ($Z$).
  - single step: $Z = \neg x \land (\exists x_0', x_1' : Z' \land TR)$
    - $Z'$ denotes OBDD $Z$ where all variables get primed ($x \to x'$)
  - if $Z$ changes, repeat previous step, otherwise fixpoint reached and computation is over

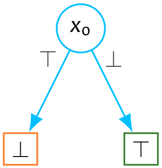
$\mathsf{EG}\,(\neg x)$ :

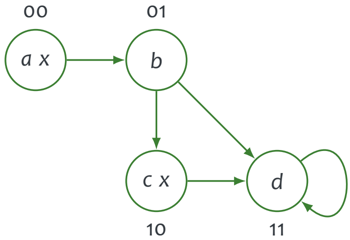


# Example

We have OBDD for states satisfying $\mathrm{EG}\,(\neg x)$ and now, we can trivially compute its negation $\neg\mathrm{EG}\,(\neg x) = \mathrm{AF}\,x$.
This corresponds to states 00 and 10 of Kripke structure.

$\neg\mathrm{EG}\,(\neg x) = \mathrm{AF}\,x$ :



# Conclusion

- During symbolic CTL model checking, all operation performed just upon OBDDs as application of logical operations and fixpoint computations.
- Usually highly efficient comparing to explicit model checking.