Decision Procedures

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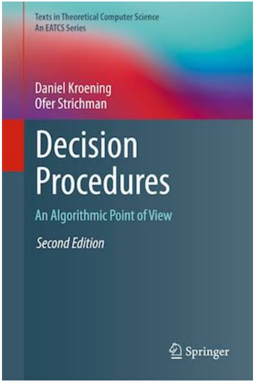
What is the course about?

- Preliminaries
- Satisfiability of boolean formulas
  - Modern SAT solvers
  - Local algorithms
  - BDDs
  - QBF
- SMT
- Decision procedures for theories
  - Equality and uninterpreted functions
  - Linear arithmetic
  - Bit vectors
  - Arrays, memory, pointers
- Combination of theories
- ...

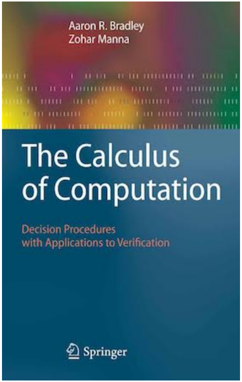
How to pass the course

- Oral exam
- Write your own SAT solver (will be explained at seminar)

Materials

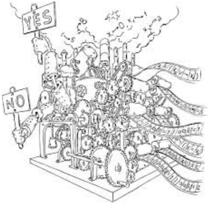


- Kroening D., Strichman O.: Decision Procedures. Second edition. Springer, 2016.



- Bradley A., Manna Z.: The Calculus of Computation. Springer, 2007.

What is a decision procedure?



Intuition

Decision procedure is an algoritm that takes a logical formula as input and decides its satisfiability.

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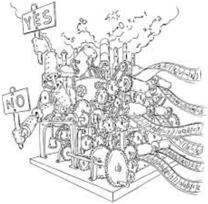

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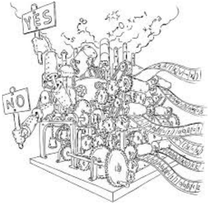

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- Satisfiable $\rightarrow$ satisfying assignment (model)
- (Unsatisfiable $\rightarrow$ proof of unsatisfiability)

Motivation

Used everywhere where logic is the primary modelling language.

- Hardware verification
  - Verifying designs of electronic circuits
- Software verification
  - Verifying that an assertion in code cannot be violated
- Compiler optimizations
  - Correctness of transformations
- Chemical reaction networks

Language of propositional logic

- Countable set of proposition variables $\{p_0, p_1, \dots\}$
- Logical connectives $\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$

Definition (Propositional formula)

1. Every propositional variable is a formula.
2. If φ, ψ are propositional formulas then also $\neg\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi$ are propositional formulas.
3. Nothing else is a propositional formula.

Basic definitions

Definition (assignment)

Assignment maps propositional variables to *true* or *false* (1 or 0, $\top$ or $\bot$)

Definition (satisfying assignment)

Formula φ is satisfied under assignment α (of its variables) if it evaluates to *true* under α. We write $\alpha \vDash \varphi$ to denote that φ is satisfied by α (α satisfies φ, α is a model of φ).

Definition (satisfiability and validity)

- A formula is satisfiable if there exists an assignment that satisfies it.
- A formula is a contradiction if it is not satisfiable.
- A formula is valid (tautology) if it is satisfied under all assignments.

Satisfiability and validity

Corollary

φ is valid iff $\neg\varphi$ is not satisfiable.

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φ *is valid iff* $\neg\varphi$ *is not satisfiable.*

If we have an algorithm for deciding satisfiability, we have an algorithm also for deciding validity (and vice versa).

Normal forms

Definition (literal, term, clause)

- *Literal* is either a propositional variable or its negation. We say literal is positive or negative, respectively.
- *Term* is a conjunction of literals.
- *Clause* is a disjunction of literals.

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Example

Let p, q be propositional variables. Then $p, \neg p, q, \neg q$ are literals, $p \wedge \neg q$ is a term, $\neg p \vee \neg q$ is a clause.

Normal forms

Definition (NNF)

A formula is in negation normal form (NNF), if it contains only $\wedge, \vee, \neg$ as connectives and negation occurs only in front of variables.

Definition (DNF)

A formula is in disjunctive normal form (DNF) if it is a disjunction of terms.

Definition (CNF)

A formula is in conjunctive normal form (CNF) if it is a conjunction of clauses.

Conversion to CNF

Lemma

For every formula there exists an equivalent formula in CNF.

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Idea of an constructive proof

- Convert to NNF.
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  - Apply De Morgan's law to propagate negation inward.
  - Apply double negation rule to eliminate double negations.
- Apply distribution law to propagate disjunction over conjunction.

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The equivalent formula can be *exponentially* larger.

Tseitin's encoding

Lemma (Tseitin)

Every formula can be converted to an equisatisfiable formula in CNF which is larger only by a constant factor.

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Idea

Introduce fresh variables to encode subformulas. Encode the meaning of these fresh variables with clauses. Avoids duplicating whole subformulas.

Tseitin's encoding

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Example: $x_1 \to x_2 \wedge x_3$

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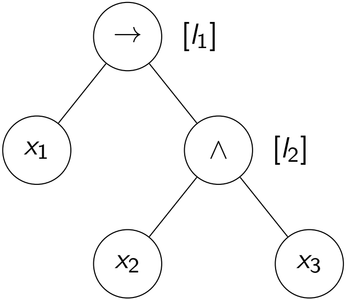

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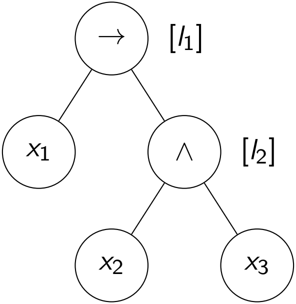

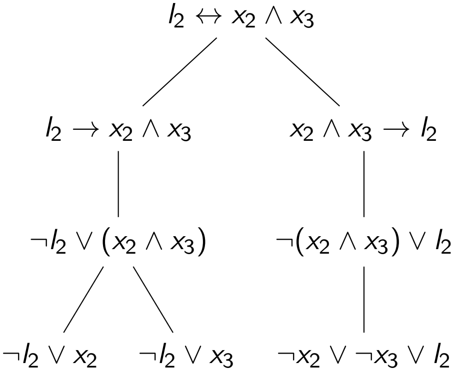

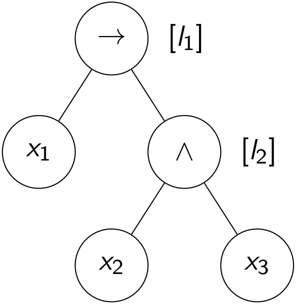

$$l_1 \leftrightarrow x_1 \rightarrow l_2$$

$$l_1 \rightarrow (x_1 \rightarrow l_2) \qquad (x_1 \rightarrow l_2) \rightarrow l_1$$

$$\neg l_1 \vee (x_1 \rightarrow l_2) \qquad (x_1 \wedge \neg l_2) \vee l_1$$

$$\neg l_1 \vee \neg x_1 \vee l_2 \qquad l_1 \vee x_1 \qquad l_1 \vee \neg l_2$$

Tseitin's encoding (formally)

Definition

Let φ be a formula and let *Repr* be a mapping of subformulas of φ to propositional variables (representants) such that:

- $Repr(p) = p$ for p a propositional variable
- $Repr(\psi) = l_\psi$ is a new unique propositional variable for non-atomic subformula.

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Let *Enc* be a mapping of subformulas to CNF formulas defined as follows:

- p a propositional variable. $Enc(p) = true$
- $\psi = \psi_1 \wedge \psi_2$. $Enc(\psi) = (\neg l_\psi \vee l_{\psi_1}) \wedge (\neg l_\psi \vee l_{\psi_2}) \wedge (l_\psi \vee \neg l_{\psi_1} \vee \neg l_{\psi_2})$
- $\psi = \psi_1 \vee \psi_2$. $Enc(\psi) = (l_\psi \vee \neg l_{\psi_1}) \wedge (l_\psi \vee \neg l_{\psi_2}) \wedge (\neg l_\psi \vee l_{\psi_1} \vee l_{\psi_2})$
- $\psi = \psi_1 \rightarrow \psi_2$. $Enc(\psi) = (l_\psi \vee l_{\psi_1}) \wedge (l_\psi \vee \neg l_{\psi_2}) \wedge (\neg l_\psi \vee \neg l_{\psi_1} \vee l_{\psi_2})$
- $\psi = \neg\psi_1$. $Enc(\psi) = (l_\psi \vee l_{\psi_1}) \wedge (\neg l_\psi \vee \neg l_{\psi_1})$

Tseitin's encoding formally

Lemma

Let φ *be a formula and let* $\varphi' = Repr(\varphi) \wedge \bigwedge_{\psi} Enc(\psi)$ *for every* ψ *a subformula of* φ*. Then* φ *and* φ' *are equisatisfiable and* $|\varphi'| = O(|\varphi|)$

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Proof.

Idea:

- Satisfying assignment of φ' restricted to the original variables is a satisfying assignment of φ
- Satisfying assignment α of φ can be extended to a satisfying assignment α' of φ' by assigning for each introduced variable l_ψ: $\alpha'(l_\psi) = \alpha(\psi)$.

□

Optimizing Tseitin's encoding

- Do not encode negative literals

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Optimizing Tseitin's encoding

- Do not encode negative literals
- Consider n-ary conjunctions and disjunctions.
- Use one-sided Tseitin's encoding for formulas in NNF.