Decision Procedures and Verification

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Algorithms for Propositional Satisfiability

Why study propositional satisfiability?

- Interesting from both theoretical and practical perspective
- First problem to be proven NP-complete [Cook '71, Levin '73]
- Many industrial problems encoded as SAT
  - Hardware and software verification
  - Automated planning - Planning as Satisfiability
  - Product configuration
  - ...

Progress in SAT solving

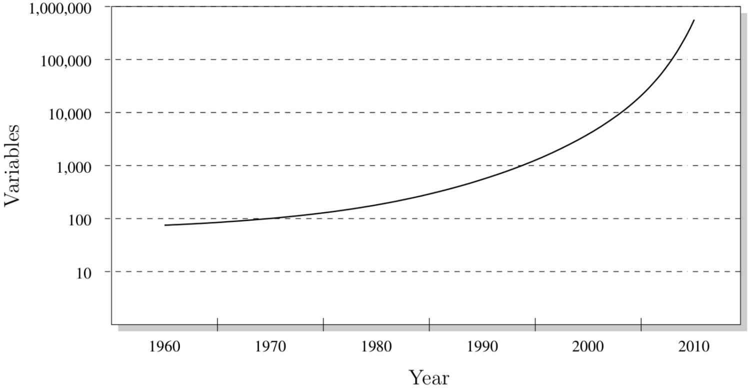

Image source: Decision Procedures. Kroening D., Strichman O.

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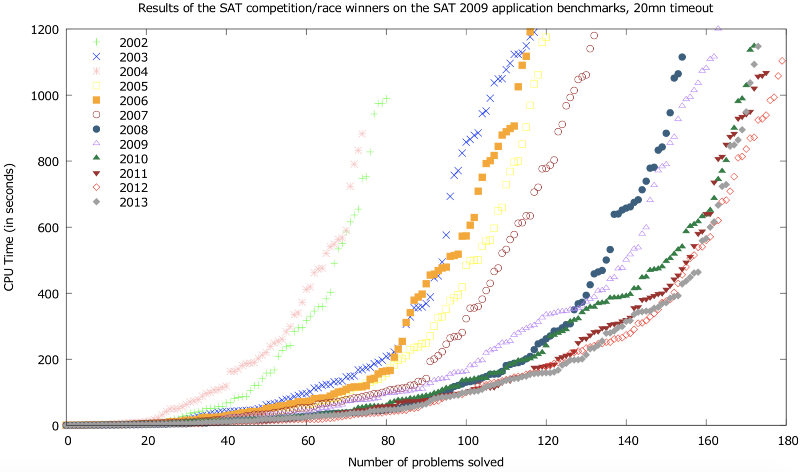

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Approaches to SAT solving

DPLL framework

- complete procedure

Stochastic search

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- complete procedure
- very efficient for instance with structure
- important when proof of unsatiafiability required (e.g. verification)

Stochastic search

- incomplete procedure
- better at solving random satisfiable instances
- can be faster to obtain satisfying assignment

Problems of naive satisfiability algorithm

Naive algorithm

Enumerate all assignments. Check if formula is satisfied under any of them.

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Naive algorithm

Enumerate all assignments. Check if formula is satisfied under any of them.

- Unnecessary repetition of partial assignment leading to conflict.
- No information preserved between tries of different assignments
- Lots of unnecessary work being done over and over again.

DPLL algorithm - overview

- *DPLL* algorithm (Davis-Putnam-Loveland-Logemann, 1962)
- Input formula assumed to be in CNF
- *Search* in a tree of partial assignments
- *Backtracking* on conflict
- *Unit propagation* prunes the tree

Basic concepts

Definition (state of a clause)

Let $\alpha : V \to \{True, False\}$ be an assignment of variables from V. Then generalization of α on a clauses of set of variables $V' \supseteq V$ is $\alpha^\star : \{c \mid c \text{ is a clause over } V'\} \to \{True, False, Undef\}$.

- c is satisfied, $\alpha^\star(c) = True$, if at least one literal in c is satisfied by α
- c is conflicting, $\alpha^\star(c) = False$, if all literals are falsified by α
- c is unresolved, $\alpha^\star(c) = Undef$, otherwise.

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- c is unresolved, $\alpha^\star(c) = Undef$, otherwise.

Definition (unit clause)

A clause c is unit under assignment α if it is not satisfied and all but one literals are falsified by α.

Basic concepts

Example

Let α be $\{x_1 \mapsto 1, x_2 \mapsto 0, x_4 \mapsto 1\}$. Then

- $x_1 \vee x_3 \vee \neg x_4$ is satisfied,
- $\neg x_1 \vee x_2$ is conflicting,
- $\neg x_1 \vee \neg x_4 \vee x_3$ is unit,
- $\neg x_1 \vee x_3 \vee x_5$ is unresolved.

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Definition (unit clause rule, antecedent clause)

Given a partial assignment α and a clause c that is unit under α, α must be extended so that is satisfies the last unassigned literal l. We say that l is implied by c (under α) and we call c the *antecedent* of l.

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Example

The clause $c = \neg x_1 \vee \neg x_4 \vee x_3$ and the partial assignment $\{x_1 \mapsto 1, x_4 \mapsto 1\}$ imply $x_3 \mapsto 1$ and $Antecedent(x_3) = c$.

The power of unit propagation

- The goal is to satisfy a CNF formula.
- $(\neg u \vee w) \wedge (u \vee v) \wedge (u) \wedge (\neg w \vee z)$; $\alpha = \{\}$
  - (u) is unit under α

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The power of unit propagation

- The goal is to satisfy a CNF formula.

- $(\neg u \vee w) \wedge (u \vee v) \wedge (u) \wedge (\neg w \vee z)$;
  $\alpha = \{u, w, z\}$
  - All clauses are satisfied by α.

  Solved by unit propagation. No decisions needed.

DPLL algorithm

procedure $\mathrm{DPLL}(\varphi, \alpha)$
    if $\forall c \in \varphi$ **then** c is satisfied by α **return** TRUE
    if $\exists c \in \varphi$ **then** c is conflicting under α **return** FALSE
    $\alpha \leftarrow \alpha \cup$ *UNIT-PROPAGATION*()
    $x \leftarrow$ *SELECT-VAR*()
    if DPLL($\alpha \cup \{x \mapsto 1\}$) **then return** TRUE
    if DPLL($\alpha \cup \{x \mapsto 0\}$) **then return** TRUE
    return FALSE

DPLL algorithm

Notes

- UNIT-PROPAGATION applies unit clause rule until no clauses are unit
- After unit propagation, assignment can be extended with *pure* literals.
  - But this is not used in practice (too costly).
- The phase of unit propagation possibly with pure literals is often referred to as *BCP* - Boolean constraint propagation
- SELECT-VAR selects an unassigned variable. Both values of the variable are tried.

DPLL - running example

$$\varphi = (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_4) \wedge (x_3 \vee x_4)$$

1. Decide $\alpha(x_1) = 1$:
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2. Derive $\alpha(x_3) = 0$:
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   $(\neg x_1 \vee \neg x_3) \wedge \underbrace{(\neg x_2 \vee \neg x_4)}_{\text{conflict clause}} \wedge (x_3 \vee x_4)$

1. Decide $\alpha(x_1) = 0$:
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DPLL - running example

$$\varphi = (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_4) \wedge (x_3 \vee x_4)$$

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$\alpha = \{x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 0\}$ is a satisfying assignment of φ

Deficiencies of DPLL

$$\psi = (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_4) \wedge (x_3 \vee x_4) \wedge (\neg y_1 \vee y_3 \vee y_4) \wedge (\neg y_2 \vee y_3 \vee y_4) \wedge (\neg y_3 \vee \neg y_4) \wedge (\neg y_3 \vee y_4) \wedge (y_3 \vee \neg y_4)$$

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- SAT solver should learn from past mistakes

Implication graph

Definition (Implication graph)

Implication graph for α is an acyclic labeled directed graph $G = (V \cup \{K\}, E)$ where:

- Vertices V correspond to variables.
  - labeled by current assignment and decision level
  - $x@N$ ($\neg x@N$): x is assigned True (False) at decision level N.
- Edges E represents *reasons* for assigning a value.
  - $(x, y) \in E$, if $\neg x \in Antecedent(y)$ with $\alpha(x) = 1$ or $x \in Antecedent(y)$ with $\alpha(x) = 0$
  - (x, y) is labeled with $Antecedent(y)$.
- Vertex K represents a conflict
  - $(x, K) \in E$, if $\neg x \in c$ with $\alpha(x) = 1$ or $x \in c$ with $\alpha(x) = 0$ where c is a conflicting clause under α.
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- *Roots* (no incoming edges) correspond to decisions, inner nodes (except K) to unit propagation. If a there is a path from roots to K we call the implication graph the *conflict* graph.

Example of implication graph

$$\varphi = (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_4) \wedge (x_3 \vee x_4)$$

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x_1@1

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$$\begin{tikzcd} x_1 @ 1 \arrow[d, "\neg x_1 \vee \neg x_3"] \\ \neg x_3 @ 1 \end{tikzcd}$$

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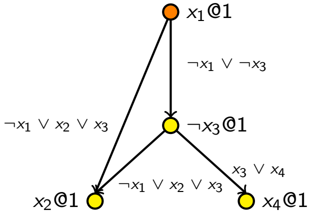

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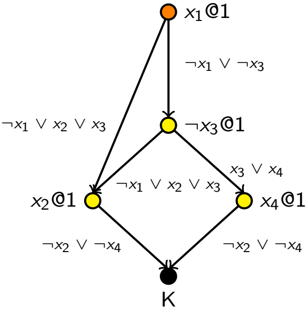
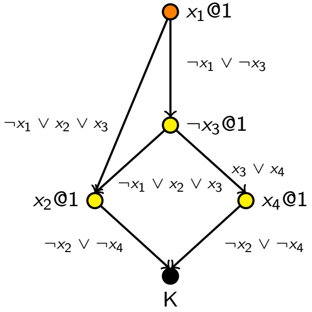


Conflict clauses as cuts in the implication graph

Definition (separating cut)

A separating cut in a conflict graph is a minimal set of edges whose removal breaks all paths from the root nodes to the conflict node.

- Each cut splits the graph to *reason* side and *conflict* side.
- The set of nodes on reason side with an edge to conflict side constitutes a sufficient condition for the conflict.
- Its negation is a conflict clause.



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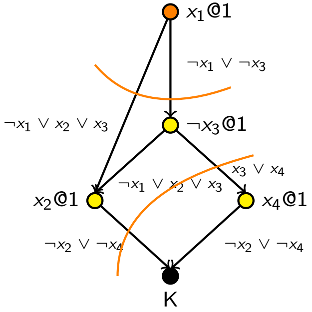
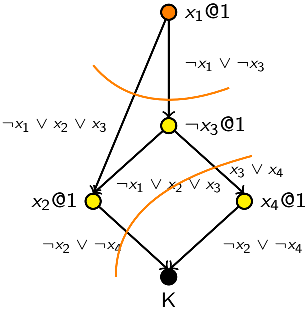


Clause learning

Observation

Every separating cut in conflict graph determines a conflict clause c such that $\varphi \to c$, where φ is the input formula.

- The conflict clause can be added to the input formula without effecting satisfiability.
- It prunes the search tree.
- This process is referred to as *learning*.
  - SAT solver is "learning" from its past mistakes.



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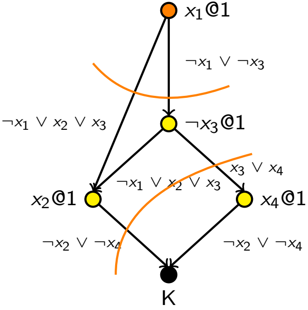

- First cut $\Rightarrow$ conflict clause $\neg x_1$.

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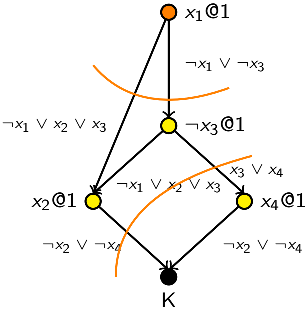

- First cut $\Rightarrow$ conflict clause $\neg x_1$.
- Second cut $\Rightarrow$ conflict clause $\neg x_2 \vee x_3$.

Clause learning strategies

- Different cuts correspond to different conflict clause.
- Impossible to predict if a clause will be more useful than other.
- In general smaller clauses are more desirable.
  - Less storage space
  - Earlier unit propagation
- Any number of conflict clauses could be learnt.
- Many SAT solvers learn a single clause with a special property, an *asserting* clause.

Asserting clause and UIP

Definition (asserting clause)

Asserting clause is a conflict clause that contains exactly one literal from the current decision level.

Definition

(unique implication point) Unique implication point (UIP) is any vertex other than K that is on all paths from the current decision level vertex to K.

- UIP always exists (at least the decision vertex itself)
- there may be more UIPs

Definition (first UIP)

First UIP is the UIP that is closest to K

Clause learning and backtracking

- Find the conflict clause containing the negation of first UIP as its single literal from current decision level.
  - asserting
- Backtracking with asserting clause
  - Backtrack to the *second highest decision level* from levels of literals in the conflict clause.
  - Equivalently (for asserting clause) to the highest decision level of its literals, excluding the UIP.
  - The newly learnt clause is unit at this decision level $\Rightarrow$ Unit propagation is immediately triggered.
- Notes:
  - If a conflict clause contains only literals from decision level 0, then the input formula is unsatisfiable.
  - If a conflict clause contains a single literal, the backtrack level is 0.

CDCL Algorithm

procedure $\mathrm{CDCL}(\varphi)$
    $\alpha \leftarrow \emptyset$
    if $BCP(\varphi, \alpha) = NULL$ **then return** FALSE
    while TRUE **do**
        $(x, v) \leftarrow SELECT(\varphi, \alpha)$
        if $(x, v) = NULL$ **then return** TRUE
        $\alpha \leftarrow \alpha \cup \{x \leftarrow v\}$
        $(result, \alpha) \leftarrow BCP(\varphi, \alpha)$
        while not result **do**
            $(level, \varphi) \leftarrow ANALYZE\text{-}CONFLICT(\varphi, \alpha)$
            if $level < 0$ **then return** FALSE
            $BACKTRACK(\varphi, level)$
            $(result, \alpha) \leftarrow BCP(\varphi, \alpha)$

CDCL Algorithm

- BCP. Performs unit propagation iteratively. Returns updated assignment and conflict indicator.
- SELECT. Selects unassigned variable and its polarity. Returns NULL if all variables are assigned.
- ANALYZE-CONFLICT. Determines backtrack level and extends φ with learned clause(s).
- BACKTRACK. Backtracks to the given decision level. Erases all assignments made after this level.

CDCL Algorithm

Notes

- Algorithm always terminates.
  - Idea of a proof: The algorithm never enters the same decision level with the same partial assignment twice.

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  - Many of them are not used more than once.

CDCL Algorithm

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- Learned clauses can be pruned.
  - Too many learned clauses slow down the solver too much.
  - Many of them are not used more than once.
- Implication graph can be represented implictly (decision trail with decision levels and polarity for variables, map of literals to antecedents).

Computing asserting clause

procedure $\mathrm{ANALYZE\text{-}CONFLICT}(\varphi, \alpha)$
    if *decision-level* $= 0$ **then return** $(-1, \varphi)$
    $c \leftarrow$ unsatisfied clause w.r.t. α
    while c is not asserting **do**
        $l \leftarrow$ most recently assigned literal in c
        $c \leftarrow RESOLVE(c, Antecedent(l), Var(l))$
    $\varphi \leftarrow \varphi \cup c$
    return $(LEVEL(c), \varphi)$

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- $RESOLVE(c_1, c_2, v)$ returns resolvent of c_1 and c_2 where x is the resolution variable.
- $LEVEL(c)$ returns the second highest decision level of literals in c. (Returns 0 is c has only one literal.)